The THEORY of BLACKJACK

Fifth Edition
Indexed

The Compleat Card Counter's
Guide to the Casino Game of 21

PETER A. GRIFFIN

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FOREWORD TO THE READER

"You have to be smart enough to understand the game and dumb enough to think it matters."
Eugene McCarthy, on the similarity between politicians and football coaches.

This book will not teach you how to play blackjack; I assume you already know how. Individuals who don't possess an acquaintance with Thorp's **Beat The Dealer**, Wilson's **Casino Gambler's Guide**, or Epstein's **Theory of Gambling and Statistical Logic** will probably find it inadvisable to begin their serious study of the mathematics of blackjack here. This is because I envision my book as an extension, rather than a repetition, of these excellent works.

Albert Einstein once said "everything should be made as simple as possible, but no simpler." For this reason I assume that all readers have an understanding of the rudiments of probability, at least to the extent of multiplying and adding appropriate fractions.

However, I recognize that the readers will have diverse backgrounds and accordingly I have divided each chapter into two parts, a main body and a subsequent, parallel, "mathematical appendix." My purpose in doing this is not to dissuade those without knowledge of advanced mathematics or statistics from reading the Appendix, but rather to forewarn them that the arguments presented may occasionally transcend their level of preparation. Thus advised, they will then be able to skim over the formulas and derivations which mean little to them and still profit quite a bit from some comments and material which just seemed to fit more naturally in the Appendices.

Different sections of the Appendices are lettered for convenience and follow the development within the chapter itself. The Appendix to Chapter One will consist of a bibliography of all books or articles referred to later. When cited in subsequent chapters only the author's last name will be mentioned, unless this leads to ambiguity.

For the intrepid soul who disregards my warning and insists on plowing forward without the slightest knowledge of blackjack at all, I have included two Supplements, the first to acquaint him with the rules, practices, and terminology of the game and the second to explain the fundamental principles and techniques of card counting. These will be found at the end of the book.

Revised Edition

On November 29, 1979, at 4:30 PM, just after the first edition of this book went to press, the pair was split for the first time under carefully controlled laboratory conditions. Contrary to original fears there was only an insignificant release of energy, and when the smoke had cleared I discovered that splitting exactly two nines against a nine yielded an expectation of precisely $-.0531$ on each of the split cards. Only minutes later a triple split of three nines was executed, producing an expectation of $-.0572$ on each card.

Development of an exact, composition dependent strategy mechanism as well as an exact, repeated pair splitting algorithm now enables me to update material in Chapters Six, Eight, and, particularly, Eleven where I present correct basic strategy recommendations for any number of decks and different combinations of rules.

There is new treatment of Atlantic City blackjack in Chapters Six and Eight. In addition the Chapter Eight analysis of Double Exposure has been altered to reflect rule changes which have occurred since the original material was written. A fuller explication of how to approximate gambler's ruin probabilities for blackjack now appears in the Appendix to Chapter Nine. A brand new Chapter Twelve has been written to bring the book up to date with my participation in the Fifth National Conference on Gambling.

Elephant Edition

In December, 1984, The University of Nevada and Penn State jointly sponsored the Sixth National Conference on Gambling and Risk Taking in Atlantic City. The gargantuan simulation results of my colleague Professor John Gwynn of the Computer Science Department at California State University, Sacramento were by far the most significant presentation from a practical standpoint and motivated me to adjust upwards the figures on pages 28 and 30, reflecting gain from computer-optimal strategy variation.

My own contribution to the conference, a study of the nature of the relation between the actual opportunity occurring as the blackjack deck is depleted and the approximation provided by an ultimate point count, becomes a new Chapter Thirteen. In this chapter the game of baccarat makes an unexpected appearance, as a foil to contrast with blackjack. Readers interested in baccarat will be rewarded with the absolutely most powerful card counting methods available for that game.

Loose ends are tied together in Chapter Fourteen where questions which have arisen in the past few years are answered. Perhaps most importantly, the strategy tables of Chapter Six are modified for use in any number of decks. This chapter concludes with two sections on the increasingly popular topic of risk minimization.

It is appropriate here to acknowledge the valuable assistance I have received in writing this book. Thanks are due to: many individuals (among whom John Ferguson, Alan Griffin, and Ben Mulkey come to mind) whose conversations helped expand my imagination on the subject; John Christopher, whose proofreading prevented many ambiguities and errors; and, finally, readers Wong, Schlesinger, Bernhardt, Gwynn, French, Wright, Early, and especially the eagle-eyed Speer for pointing out mistakes in the earlier editions. Photographic credits go to Howard Schwartz, John Christopher, Marcus Marsh, and the Sacramento Zoo.

To John Luckman

"A merry old soul was he"
Las Vegas will miss him,
and so will I.

1

INTRODUCTION

"There are three subjects you can count upon a man to lie about: sex, gas mileage, and gambling."
R. A. Rosenbaum

I played my first blackjack in January, 1970, at a small club in Yerington, Nevada. Much to the amusement of a local Indian and an old cowboy I doubled down on (A,9) and lost. No, it wasn't a knowledgeable card counting play, just a beginner's mistake, for I was still struggling to learn the basic strategy as well as fathom the ambiguities of the ace in "soft" and "hard" hands. The next day, in Tonopah, I proceeded to top this gaffe by standing with (5,4) against the dealer's six showing; my train of thought here had been satisfaction when I first picked up the hand because I remembered what the basic strategy called for. I must have gotten tired of waiting for the dealer to get around to me at the crowded table since, after the dealer made 17 and turned over my cards, there, much to everyone's surprise, was my pristine total of nine!

At the time, I was preparing to give a course in The Mathematics of Gambling which a group of upper division math majors had petitioned to have offered. It had occurred to me, after agreeing to teach it, that I had utterly no gambling experience at all; whenever travelling through Nevada with friends I had always stayed outside in the casino parking lot to avoid the embarrassment of witnessing their foolishness.

But now I had an obligation to know first hand about the subject I was going to teach. An excellent mathematical text, R. A. Epstein's **Theory of Gambling and Statistical Logic**, had come to my attention, but to adequately lead the discussion of our supplementary reading, Dostoyevsky's **The Gambler**, I clearly had to share this experience.

At first I had no particular interest in card counting or blackjack, but after totalling up the losses of my brief, between semesters, novitiation, I vowed revenge on the casinos. What the text informed me was that, short of armed robbery or counterfeiting chips (and I had considered these), there was only one way to get my money back. With this in mind I anted up $1.95 for Ed Thorp's classic, **Beat the Dealer,** which even today at $2.45 I still consider the best buy on the subject.

Soon, indeed, I had recouped my losses and was playing with their money, but it wasn't long before the pendulum swung the other way again. Although this book should prove interesting to those who hope to profit from casino blackjack, I can offer them no encouragement, for today I find myself farther behind in the game than I was after my original odyssey in 1970. I live in dread that I may never again be able to even the score, since it may not be possible to beat the hand held game and four decks bore me to tears.

My emotions have run the gamut from the inebriated elation following a big win which induced me to pound out a chorus of celebration on the top of an occupied Reno police car to the frustrated depths of biting a hole through a card after picking up what seemed my 23rd consecutive stiff hand against the dealer's ten up card. I've stared at the ceiling in the mockingly misnamed Victory Motel, wondering how in the name of Probability I could be good enough to win $400 in six hours of steady play downtown and bad enough to then lose $100 in each of nine Las Vegas Strip casinos in only three and a half hours that evening.

My playing career has had a sort of a Faustian aspect to it, as I began to explore the mysteries of the game I began to lose, and the deeper I delved, the more I lost. There was even a time when I wondered if Messrs. Thorp, Wilson, Braun, and Epstein had, themselves, entered into a pact with the casinos to deliberately exaggerate the player's odds in the game. But after renewing my faith by confirming their figures for the basic game, I threw myself once again into the fray, alas with the same results.

Why This Book?

Why then should I presume to write a book on this subject? Perhaps, like Stendahl, "I prefer the pleasure of writing all sorts of foolishness to that of wearing an embroidered coat costing 800 francs." Certainly if I did have some secret to riches I wouldn't share it with the public until I was thoroughly sated myself. But I do have a knowledge of the theoretical probabilities to share with those who are interested; unfortunately my experience offers no assurance that these will be realized, in the short or the long run.

To extend G.B. Shaw's insight: If you can do something, then you do it; if you can't, you teach others to do it; if you can't teach, you teach people to teach; and if you can't do that, you administrate. I must, I fear, like Marx, relegate myself to the role of theoretician rather than active revolutionary. Long since disabused of the notion that I can win a fortune in the game, my lingering addiction is to the pursuit of solutions to the myriad of mathematical questions posed by this intriguing game.

Difficulty interpreting Randomness

My original attitude of disapproval towards gambling has been mitigated somewhat over the years by a growing appreciation of the possible therapeutic benefits from the intense absorption which overcomes the bettor when awaiting the verdict of Lady Luck. Indeed, is there anyone who, with a wager at stake, can avoid the trap of trying to perceive patterns when confronting randomness, of seeking "purpose where there is only process?"* Our entire education is in the direction of trying to make sense out of our environment; as a result we often experience our greatest difficulties trying to understand that which has utterly no meaning.

Not long ago a Newsweek magazine article described Kirk Kerkorian as "an expert crapshooter." I am intrigued to learn what it is that distinguishes the expert from the novice in a series of negative expectation guesses on the results of independent trials. Nevertheless, while we can afford to be a bit more sympathetic to those who futilely try to impose a system on dice, keno, or roulette, we should not be less impatient in urging them to turn their attention to the dependent trials of blackjack.

Blackjack's Uniqueness

This is because blackjack is unique among all casino games in that it is a game in which skill should make a difference, even swing the odds in the player's favor. Because of

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*Kamongo, by Homer Smith

the possibilities of using information and exercising rational choice, this game has an appeal to many who wouldn't ordinarily be interested in gambling. Some will also enjoy the game for its solitaire-like aspect; since the dealer has no choices it's like batting a ball against a wall; there is no opponent and the collisions of ego which seem to characterize so many games of skill, like bridge and chess, do not occur.

Use of Computers

Ultimately, all mathematical problems related to card counting are Bayesian; they involve conditional probabilities subject to information provided by a card counting parameter. It took me an inordinately long time to realize this when I was pondering how to find the appropriate index for insurance with the Dubner HiLo system.

Following several months of wasted bumbling I finally realized that the dealer's conditional probability of blackjack could be calculated for each value of the HiLo index by simple enumerative techniques. My colleague, Professor John Christopher, wrote a computer program which provided the answer and also introduced me to the calculating power of the device. To him I owe a great debt for his patient and priceless help in teaching me how to master the machine myself. More than once when the computer rejected or otherwise played havoc with one of my programs he counseled me to look for a logical error rather than to persist in my demand that an electrician be called in to check the supply of electrons for purity.

After this first problem, my interest became more general. Why did various count strategies differ occasionally in their recommendations on how to play some hands? What determined a system's effectiveness anyway? How good were the existing systems? Could they be measureably improved, and if so, how?

Although computers are a *sine qua non* for carrying out lengthy blackjack calculations, I am not as infatuated by them as many of my colleagues in education. It's quite fashionable these days to orient almost every course toward adaptability to the computer. To this view I raise the anachronistic objection that one good Jesuit in our schools will accomplish more than a hundred new computer terminals. In education the means is the end; how facts and calculations are produced by our students is more important than how many or how precise they are.

One of the great dreams of a certain segment of the card counting fraternity is to have an optimal strategy computer at their disposal for actual play. Fascinated by Buck Rogers gadgetry, they look forward to wiring themselves up like bombs and stealthily plying their trade under the very noses of the casino personnel, fueled by hidden power sources.

For me this removes the element of human challenge. The only interest I'd have in this machine (a very good approximation to which could be built with the information in Chapter Six of this book) is in using it as a measuring rod to compare how well I or others could **play** the game. Indeed one of the virtues I've found in not possessing such a contraption, from which answers come back at the press of a button, is that, by having to struggle for and check approximations, I've developed insights which I otherwise might not have achieved.

Cheating

No book on blackjack seems complete without either a warning about, or whitewashing of, the possibility of being cheated. I'll begin my comments with the frank admission that I am completely incapable of detecting the dealing of a second, either by sight or by sound. Nevertheless I know I have been cheated on some occasions and find myself wondering just how often it takes place. The best card counter can hardly expect to have more than a two percent advantage over the house; hence if he's cheated more than one hand out of fifty he'll be a loser.

I say I **know** I've been cheated. I'll recite only the obvious cases which don't require proof.

I lost thirteen hands in a row to a dealer before I realized she was deliberately interlacing the cards in a high low stack.

Another time I drew with a total of thirteen against the dealer's three; I thought I'd busted until I realized the dealer had delivered two cards to me: the King that broke me and, underneath it, the eight she was clumsily trying to hold back for herself since it probably would fit so well with her three.

I had a dealer shuffle up twice during a hand, both times with more than twenty unplayed cards, because she could tell that the card she just brought off the deck would have helped me: "Last card" she said with a quick turn of the wrist to destroy the evidence.

In another recent episode a dealer always seemed to take an inordinate amount of time waiting for the players to insure. Then she either didn't or did have blackjack depending it seems, on whether they did or didn't insure; unfortunately the last time when she turned over her blackjack there was also a four hiding underneath with the ten!

As I mentioned earlier, I had been moderately successful playing until the "pendulum swung." Trying to discover some reason for Dame Fortune's fickleness, I embarked upon a lengthy observation of the frequency of dealer up cards in the casinos I had suffered most in. The result of my sample, that the dealers had 770 tens or aces out of 1820 hands played, was a statistically significant indication of some sort of legerdemain. However, you are justified in being reluctant to accept this conclusion since the objectivity of the experimenter can be called into question; I produced evidence to explain my own long losing streak as being the result of foul play, rather than my own incompetence.

An investigator for the Nevada Gaming Commission admitted point blank at the 1975 U. of Nevada Gambling Conference, that the customer was liable to be cheated in the "cow towns", but he echoed the usual refrain that the big clubs have too much to lose to allow it to happen there. I find little solace in this view that Nevada's country bumpkins are less trustworthy but more dextrous than their big city cousins. I am also left wondering about the responsibility of the Gaming Commission since, if they knew the allegation was true why didn't they close the places, and if they didn't, why would their representative have made such a statement?

One of the overlooked motivations for a dealer to cheat is not financial at all, but psychological. The dealer is compelled by the rules to function like an automaton and may be inclined, either out of resentment toward someone (the card counter) doing something of which he's incapable or out of just plain boredom, to substitute his own determination for that of fate. Indeed, I often suspect that many dealers who can't cheat like to suggest they're in control of the game by cultivation of what they imagine are the mannerisms of a card-sharp. The best cheats, I assume, have no mannerisms.

Are Card Counters Cheating?

Credit for one of the greatest brain washing achievements must go to the casino industry for promulgation of the notion that card counting itself is a form of cheating. Not just casino employees, but many members of the public, too, will say: "tsk, tsk, you're not supposed to keep track of the cards", as if there were some sort of moral injunction to wear blinders when entering a casino.

Robbins Cahill, director of the Nevada Resort Association, was quoted in the Las Vegas Review of August 4, 1976 as saying that most casinos "don't really like the card counters because they're changing the natural odds of the game."

Nonsense. Card counters are no more changing the odds than a sunbather alters the weather by staying inside on rainy days! And what are these "natural odds"? Do the casinos dislike the player who insures a pair of tens against the dealer's ace and then splits them repeatedly, rejecting any total of twenty he is dealt? Is not this, too, as "unnatural" an act as standing on (4,4,4,4) against the dealer's ten after you've seen another player draw four fives? Somehow the casinos would have us believe the former is acceptable but the latter is ethically suspect.

It's certainly understandable that casinos do not welcome people who can beat them at their own game; particularly, I think, they do not relish the reversal of roles which takes place where they become the sucker, the chump, while the card counter becomes the casino, grinding them down. The paradox is that they make their living encouraging people to believe in systems, in luck, cultivating the notion that some people are better gamblers than others, that there is a savvy, macho personality that can force dame fortune to obey his will.

How much more sporting is the attitude of our friends to the North! Consider the following official policy statement of the Province of Alberta's Gaming Control Section of the Department of the Attorney General:

"Card counters who obtain an honest advantage over the house through a playing strategy do not break any law. . . Gaming supervisors should ensure that no steps are taken to discourage any player simply because he is winning."

So remember now, players and dealers both, from now on, no cheating; it makes the mathematics too untidy.

APPENDIX TO CHAPTER 1

It seems appropriate to list here, as well as comment on, all the works which will be referred to subsequently. Books of a less technical nature I deliberately do not mention. There are many of these, of varying degrees of merit, and one can often increase his general awareness of blackjack by skimming even a bad book on the subject, if only for the exercise in criticism it provides. However, reference to any of them is unnecessary for my purposes and I will confine my bibliography to those which have been of value to me in developing and corroborating a mathematical theory of blackjack.

BIBLIOGRAPHY

ANDERSON, T. *An Introduction to Multivariate Statistical Analysis*, Wiley, 1958. This is a classical reference for multivariate statistical methods, such as those used in Chapter Five.

BALDWIN, CANTEY, MAISEL, and McDERMOTT. *Journal of the American Statistical Association*, Vol. 51, 419-439; 1956 This paper is the progenitor of all serious work on blackjack. It is remarkably accurate considering that the computations were made on desk calculators. Much of their terminology survives to this day.

BALDWIN, et alii. *Playing Blackjack to Win*, M. Barrons and Company, 1957. This whimsical, well written guide to the basic strategy also contains suggestions on how to vary strategy depending upon cards observed during play. This may be the first public mention of the possibilities of card counting. Unfortunately it is now out of print and a collector's item.

BRAUN, Julian *The Development and Analysis of Winning Strategies for Casino Blackjack*, private research report. Braun presents the results of several million simulated hands as well as a meticulous explanation of many of his computing techniques.

EPSTEIN, R.A. *Theory of Gambling and Statistical Logic.* New York: Academic Press, rev. 1977. In his blackjack section Epstein has an excellent treatment of how to determine basic strategy. There is also a complete version of two different card counting strategies and extensive simulation results for the ten count. What is here, and not found anywhere else, is the extensive table of player expectations with each of the 550 initial two card situations in blackjack for single deck play. There is a wealth of other gambling and probabilistic information, with a lengthy section on the problem of optimal wagering.

ERDŐS and RENYI. *On the Central Limit Theorem for Samples from a Finite Population.* Matem Kutato Intezet. Kolzem., Vol. 4, p. 49. Conditions are given to justify asymptotic normality when sampling without replacement. It is difficult to read in this untranslated version, and even more difficult to find. Better try . . .

FISZ. *Probability Theory and Mathematical Statistics.* Wiley, 1963. Exercise 14.8 on page 523 is based on the Erdos and Renyi result.

GORDON, Edward. *Optimum Strategy in Blackjack.* Claremont Economic Papers; Claremont, Calif. January 1973. This contains a useful algorithm for playing infinite deck blackjack.

GWYNN and SERI. *Experimental Comparison of Blackjack Betting Systems.* Paper presented to the Fourth Conference on Gambling, Reno, 1978, sponsored by the University of Nevada. People who distrust theory will have to believe the results of Gwynn's tremendous simulation study of basic strategy blackjack with bet variations, played on his efficient "table driven" computer program.

HEATH, David. *Algorithms for Computations of Blackjack Strategies,* presented to the Second Conference on Gambling, sponsored by the University of Nevada, 1975. This contains a good exposition of an infinite deck computing algorithm.

MANSON, BARR, and GOODNIGHT. *Optimum Zero Memory Strategy and Exact Probabilities for 4-Deck Blackjack.* The American Statistician 29(2):84-88. 1975. The authors, from North Carolina St. University, present an intriguing and efficient recursive method for finite deck blackjack calculations, as well as a table of four deck expectations, most of which are exact and can be used as a standard for checking other blackjack programs.

THORP, E.O. *Beat the Dealer.* New York: Vintage Books, 1966. If I were to recommend one book, and no other, on the subject, it would be this original and highly successful popularization of the opportunities presented by the game of casino blackjack.

THORP, E.O. *Optimal Gambling Systems for Favorable Games.* Review of the International Statistics Institute, Vol. 37:3, 1969. This contains a good discussion of the gambler's ruin problem, as well as an analysis of several casino games from this standpoint.

THORP, E.O. and WALDEN, W.E. *The Fundamental Theorem of Card Counting.* International Journal of Game Theory, Vol. 2, 1973, Issue 2. This paper, presumably an outgrowth of the authors' work on baccarat, is important for its combinatorial demonstration that the spread, or variation, in player expectation for any fixed strategy, played against a diminishing and unshuffled pack of cards, must increase.

WILSON, Allan. *The Casino Gambler's Guide.* Harper & Row, 1965. This is an exceptionally readable book which lives up to its title. Wilson's blackjack coverage is excellent.

In addition, any elementary statistics text may prove helpful for understanding the probability, normal curve, and regression theory which is appealed to. I make no particular recommendations among them.

2

THE BASIC STRATEGY

When I had an ace and jack
I heard a wise man say,
"Give crowns and pounds and guineas
But not your natural away;
Give pearls away and rubies
But let your two win three."
But then I had an ace and jack,
No use to talk to me.

When I had an ace and jack
I heard him say again,
"If you draw another card
It will not be a ten;
You'll wish you hadn't doubled
And doubtless you will rue."
Now I have ace, jack, and two
And Oh, 'tis true, 'tis true.

Shameless Plagiarism of A. E. Housman

Unless otherwise specified, all subsequent references will be to single deck blackjack as dealt on the Las Vegas Strip: dealer stands on soft 17, player may double on any two initial cards, but not after splitting pairs. Furthermore, although it is contrary to almost all casino practices, it will be assumed, when necessary to illustrate general principles of probability, that all 52 cards will be dealt before reshuffling.

The first questions to occur to a mathematician when facing a game of blackjack are: (1) How should I play to maximize my expectation? and (2) What is that maximal expectation? The answer to the first determines the answer to the second, and the answer to the second determines whether the mathematician is interested in playing.

Definition of Basic Strategy

The *basic strategy* is the strategy which maximizes the player's average gain, or expectation, playing one *hand* against a complete pack of cards. Thus, with a given number of decks and fixed set of rules there can be only one "basic strategy," although there may be several (slightly erroneous) versions of it. It is even conceivable, if not probable, that nobody, experts included, knows precisely what the basic strategy is, if we pursue the definition to include instructions on how to play the second and subsequent cards of a split depending on what cards were used on the earlier parts. For example, suppose we split eights against the dealer's ten, busting the first hand (8,7,7) and reaching (8,2,2,2) on the second. Quickly now, do we hit or do we stand with the 14? (You will be able to find answers to such questions after you have mastered Chapter Six.)

The basic strategy, then, constitutes a complete set of decision rules covering all possible choices the player may encounter, but without any reference to any other players' cards or any cards used on a previous round before the deck is reshuffled. These choices are: to split or not to split, to double down or not to double down, and to stand or to draw another card. Some of them seem self evident, such as always drawing another card to a total of six, never drawing to twenty, and not splitting a pair of fives. But what procedure must be used to assess the correct action in more marginal cases?[A]

Hitting and Standing

As an example consider the choice of whether to draw or stand with (T,6)* against a dealer 9. While relatively among the simplest borderline choices to analyze, we will see that precise resolution of the matter requires an extraordinary amount of arithmetic.

If we stand on our 16, we will win or lose solely on the basis of whether the dealer busts; there will be no tie. The dealer's exact chance of busting can be found by pursuing all of the 566 distinguishably different drawing sequences and weighting

*The letter T will be adopted as a symbol for any ten-valued card, whether 10,Jack,Queen, or King.

their paths according to their probability of occurrence. A few such sequences which lead to a bust are:

Drawing Sequence	Probability
9,2,A,T	4/49x4/48x15/47
9,7,6	4/49x3/48
9,2,A,A,A,A,8	4/49x4/48x3/47x2/46x1/45x4/44

Obviously a computer will be necessary to carry out the computations with satisfactory accuracy and speed. In Chapter Eleven there will be found just such a program.

Once the deed has been done we find the dealer's exact chance of busting is .2304, and it is time to determine the "mathematical expectation" associated with this standing strategy. Since we win .2304 bets for every .7696 ones we lose, our average return is .2304 − .7696 = −.5392, which has the interpretation that we "expect" to lose 54 cents on the dollar by hoping the dealer will break and not risking a bust ourselves.

This has been the easy part; analysis of what happens when we draw a card will be more than fivefold more time consuming. This is because, for each of the five distinguishably different cards we can draw without busting (A,2,3,4,5), the dealer's probabilities of making various totals, and not just of busting, must be determined separately.

For instance, if we draw a two we have 18 and presumably would stand with it. How much is this hand of T+6+2=18 worth, or in mathematician's language, what is our conditional expectation *if* we get a two when drawing? We must go back to our dealer probability routine and play out the dealer's hand again, only now from a 48 card residue (our deuce is unavailable to the dealer) rather than the 49 card remainder used previously. Once this has been done we're interested not just in the dealer's chance of busting, but also specifically in how often he comes up with 17,18,19,20,and 21. The result is found in the third line of the next table.

DEALER'S CHANCES WHEN SHOWING 9

Player's Cards	17	18	19	20	21	Bust
T,6						.2304
T,6,A	.1259	.1093	.3576	.1076	.0636	.2360
T,6,2	.1248	.1045	.3553	.1265	.0565	.2324
T,6,3	.1244	.1060	.3532	.1265	.0621	.2278
T,6,4	.1257	.1054	.3546	.1243	.0619	.2281
T,6,5	.1252	.1060	.3549	.1256	.0598	.2285

With our (T,6,2), or 18, we will win .1248 + .2324 = .3572, and lose .3553 + .1265 + .0565 = .5383. Hence our "conditional expectation" is .3572 − .5383 = −.1811. Some readers may be surprised that a total of 18 is overall a losing hand here. Note also that the dealer's chance of busting increased slightly, but not significantly, when he couldn't use "our" deuce.

Similarly we find all other conditional expectations.

Player draws	for total of	with probability	and consequent expectation of	to contribute
A	17	4/49	-.4021	-.0328
2	18	"	-.1811	-.0148
3	19	"	.2696	.0220
4	20	"	.7519	.0614
5	21	"	.9402	.0768
bust card	too much	29/49	-1.0000	-.5918
			Bottom line is	-.4793

In the column labeled "to contribute" we multiply each of the expectations by its probability; the total of this column, or "bottom line," is our expectation if we draw a card. Since a loss of 48 cents by drawing is preferable to one of 54 cents from standing, basic strategy is to draw to (T,6) v 9. Note that it was assumed that we would not draw a card to (T,6,A), (T,6,2), etc. This decision would rest on a previous and similar demonstration that it was not in our interest to do so.

Analysis of our best strategy and consequent expectation with a smaller total of possibly more than two cards, such as (5,4,3), would be based on a sort of recursive reference to previous calculations of our optimal expectation and strategy with (5,4,3,A), (5,4,3,2), . . . (5,4,3,9), (5,4,3,A,A), etc. All this is very tedious and time consuming, but necessary if the *exact* player expectation is sought. This, of course, is what computers were designed for; limitations on the human life span and supply of paper preclude an individual doing the calculations by hand.

Doubling Down

So much for the choice of whether to hit or stand in a particular situation, but how about the decision on whether to double down or not? In some cases the decision will be obviously indicated by our previous calculations, as in the following example.

Suppose we have (A,6) v dealer 5. (Any two card total of hard 10 or 11 would illustrate the situation equally well against the dealer's up card of 5.) We know three things:

1. We want to draw another card, it having already been determined that drawing is preferable to standing with soft 17.
2. We won't want a subsequent card no matter what we draw (for instance, drawing to (A,6,5) would be about 7% worse than standing).
3. Our overall expectation from drawing one card is positive—that is, we have the advantage.

Hence the decision is clear; by doubling down we make twice as much money as by conducting an undoubled draw.

The situation is not quite so obvious when contemplating a double of (8,2) v 7. Conditions 1 and 3 above still hold, but if we receive a 2,3,4,5, or 6 in our draw we would like to draw another card, which is not permitted if we have selected the double down option. Therefore, we must compare the amount we lose by forfeiting the right to draw another card with the amount gained by doubling our bet on the one card draw. It turns out we give up about 6% by not drawing a card to our subsequently developed stiff hands, but the advantage on our extra, doubled, dollar is 21%. Since our decision to *double* raises expectation it becomes part of the basic strategy.

The Baldwin group pointed out in their original paper that most existing recommendations at the time hardly suggested doubling at all. Probably the major psychological reason for such a conservative attitude is the sense of loss of control of the hand, since another card cannot be requested. Doubling on small soft totals, like (A,2), heightens this feeling, because one could often make a second draw to the hand with no risk of busting whatsoever. But enduring this sense of helplessness, like taking a whiff of ether before necessary surgery, is sometimes the preferable choice.

Pair Splitting

Due to their infrequency of occurrence, decisions about pair splitting are less important, but unfortunately much more complicated to resolve. Imagine we have (7,7)v9. The principal question facing us is whether playing one fourteen is better than playing two, or more, sevens in what is likely to be a losing situation.

Determination of the exact splitting expectation requires a tortuous path. First, the exact probabilities of ending up with two, three, and four sevens would be calculated. Then the player's expectation starting a hand with a seven in each of the three cases would be determined by the foregoing methods. The overall expectation would result from adding the product of the probabilities of splitting a particular number of cards and the associated expectations. The details are better reserved for Chapter Eleven, where a computer procedure for pair splitting is outlined.

Summing Up

Finally, the player's total expectation for basic strategy blackjack is obtained as a weighted sum of all $55 \times 10 = 550$ expectations calculated for each of the 55 different player hands and 10 different dealer up cards. An abridgement of the necessary ledger is

Player Hand	Dealer Up Card	Probability	Expectation	Product
T,6	9	64/1326 x 4/50	-.4793	-.00185
A,6	5	16/1326 x 4/50	.2800	.00027
8,2	7	" " "	.4166	.00040
7,7	9	6/1326 x 4/50	-.4746	-.00017

The remarkable thing is that the bottom line, or net result of the entire calculation, turns out to be exactly zero when rounded off to the nearest tenth of a percent. This, of course, is for the set of rules and single deck we assumed. It's not inconceivable that this highly complex game is closer to the mathematician's ideal of "a fair game" (one which has zero expectation for both competitors) than the usually hypothesized coin toss, since real coins are flawed and might create a greater bias than the fourth decimal of the blackjack expectation, whatever it may be.

Condensed Form of the Basic Strategy

By definition, the description of the basic strategy is "composition" dependent rather than "total" dependent in that some card combinations which have the same total, but unlike compositions, require a different action to optimize expectation. This is illustrated by considering two distinct three card 16's to be played against the dealer's Ten as up card: with (7,5,4) the player is 4.7% better off standing, while with (6,4, 6) he gains 2.3% by hitting.

Notwithstanding these many "composition" dependent exceptions (which tax the memory and can be ignored at a total cost to the player of at most .04%) we'll define a "total" dependent basic strategy, recognizing all the while that it is a simplification for convenience of reference.[B]

Hard drawing and standing:	Never hit 17 or higher. Hit stiff totals (12 to 16) against high cards (7,8,9,T,A), but stand with them against small cards (2,3,4,5,6), except hit 12 against a 2 or 3.
Soft drawing and standing:	Always draw to 17 and stand with 18, except hit 18 against 9 or T.
Pair Splitting:	Never split (4,4), (5,5), or (T,T), but always split (8,8) and (A,A). Split (9,9) against 2 through 9, except not against a 7. Split the others against 2 through 7, except hit (6,6) v 7, (2,2) and (3,3) v 2, and (3,3) v 3.
Hard Doubling:	Always double 11. Double 10 against all cards except T or A. Double 9 against 2 through 6. Double 8 against 5 and 6.
Soft Doubling:	Double 13 through 18 against 4,5, and 6. Double 17 against 2 and 3. Double 18 against 3. Double 19 against 6.

House Advantage

If you ask a casino boss how the house derives its advantage in blackjack he will probably reply "The player has to draw first and if he busts, we win whether we do or not." This fact might escape a rube in Reno with a few coins jingling in his pocket. Being ignorant of our basic strategy, such an individual's inclination might not unnaturally be to do what the Baldwin group aptly termed "Mimicking the Dealer"—that is hitting all his hands up to and including 16 without any discrimination of the dealer's up card.

This "mimic the dealer" strategy would give the house about a 5.5% edge since dealer and player would both break with probability 28%. Thus the "double bust," which provides the house with the embryo of whatever advantage it enjoys, would occur about 28% of 28%, or 8%, of the time. Since all other situations would symmetrize, this seems to put the disadvantage at 8%, but that is to ignore the almost one chance in twenty when the player gets a blackjack and receives an extra half dollar that the dealer doesn't get.

How can the basic strategist whittle this 5.5% down to virtually nothing? The following chart of departures from "mimic the dealer" is a helpful way to understand the nature of the basic strategy.

DEPARTURES FROM "MIMIC THE DEALER"

Option	Gain
Proper pair splitting	.4%
Doubling down	1.6%
Hitting soft 17,18	.3%
Proper standing	3.2%

Thus we see the doubling, splitting and standing decisions are crucial and the best way to gain insight into some of them is to look at a chart of the dealer's busting probabilities.

Up Card	2	3	4	5	6	7	8	9	T	A
% Chance of Bust	35	38	40	43	42	26	24	23	21	11

Note that most of the aggressive actions, like doubling and splitting, are taken when the dealer shows a small card, and these cards bust most often overall, about 40% of the time. Incidentally, I feel the quickest way to determine if somebody is a bad player is to watch whether his initial eye contact is with his own, or the dealer's first card. The really unskilled function as if the laws of probability had not yet been discovered and seem to make no distinction between a five and an ace as dealer up card.

APPENDIX TO CHAPTER 2

A.

The interested reader can profit by consulting several other sources about the mathematics of basic strategy. Wilson has a lengthy section on how he approached the problem, as well as a unique and excellent historical commentary about the various attempts to assess the basic strategy and its expectation. The Baldwin group's paper is interesting in this light.

Manson *et alii* present an almost exact determination of 4 deck basic strategy, and it is from their paper that I became aware of the exact recursive algorithm they use. They credit Julian Braun with helping them, and I'm sure some of my own procedures are belated germinations of seeds planted when I read various versions of his monograph.

Infinite deck algorithms were presented at the First and Second Gambling Conferences, respectively by Edward Gordon and David Heath. These, of course, are totally recursive. Their appeal stems ironically from the fact that it takes far less time to deal out all possible hands from an infinite deck of cards than it does from one of 52 or 208!

B.

The two card, "composition" dependent, exceptions are standing with (7,7) v T, standing with (8,4) and (7,5) v 3, hitting (T,2) v 4 and 6, hitting (T,3) v 2, and not doubling (6, 2) v 5 and 6. The multiple card exceptions are too numerous to list, although most can be deduced from the tables in Chapter Six.

The decision not to double (6,2) v 5 must be the closest in basic strategy blackjack. The undoubled expectation is .130630, while doubling yields .130583.

3

THE SPECTRUM OF OPPORTUNITY

"Good Grief. The poor blackjack deck is being stripped naked of all her secrets."
—Richard Epstein

When more than one hand is dealt before reshuffling, the basic strategist will realize exactly the same overall expectation on the second hand as on the first. This is easily proven by imagining all possible permutations of the deck and recognizing that, for any first and second hand that can occur, there is an equiprobable reordering of the deck which merely interchanges the two. For example, it is just as likely that the player will lose the first hand (7,5) to (6,A) and push the second (9,8) to (3,4,T) as that he pushes the first one (9,8) to (3,4,T) and loses the next (7,5) to (6,A).

If resplitting pairs were prohibited there would always be enough cards for four hands before reshuffling and that would guarantee an identical expectation for basic strategy play on all four hands. Unfortunately, with multiple splitting permitted, there is an extraordinarily improbable scenario which exhausts the deck before finishing the third hand and denies us the luxury of asserting the third hand will have *precisely* the same expectation as the first two: on the first hand split (6, 5, T), (6, 5, T), (6, 5, T), and (6, 5, T) versus dealer (2, 4, T, T); second hand, split (3, 9, 4, T), (3, 9, 4, T), (3, 9, 4, T), and (3, A, T, 7) against dealer (7, 9, T); finally, develop (8, 7, T), (8, 7, T), (8, 2, 2, A, A, A, T), and unfinished (8, 2, ?) in the face of dealer's (T, T).

Gwynn's simulation study showed no statistically significant difference in basic strategy expectation among the first seven hands dealt from a full pack and only three times in 8,000,000 decks was he unable to finish four hands using 38 cards. Thus, as a matter of practicality, we may assume the first several hands have the same basic strategy expectation.

Although it may seem contradictory, it is also true that no *particular* subsequent hand, before it is dealt, would have the same set of conditional probabilities attached to it as the first hand from a full deck. This realization leads us to consider what Thorp and Walden termed the "spectrum of opportunity" in their paper **The Fundamental Theorem of Card Counting** wherein they proved that the variations in player expectation for a fixed strategy must become increasingly spread out as the deck is depleted.

An Example

As an extreme but graphic illustration, as well as a review of the principles explicated in Chapter Two, let's consider a five card remainder which consists of [5,6,8,9,T]. Notice there are no pair splits possible and the 38 total pips available guarantee that all hands can be resolved without reshuffling. The basic strategist, while perhaps unaware of this composition, will have an expectation of 6.67% as the following exhaustive table of all 30 player-dealer situations indicates.

Player Hand	Dealer Up Card	Expectation	Player Hand	Dealer Up Card	Expectation
5,6	8	+2	6,9	5	+1
	9	+1		8	0
	T	+1		T	0
5,8	6	+1	6,T	5	+1
	9	- .5		8	0
	T	- .5		9	0
5,9	6	+1	8,9	5	-1
	8	0		6	-1
	T	0		T	-1
5,T	6	+1	8,T	5	-1
	8	0		6	-1
	9	0		9	-1
6,8	5	+1	9,T	5	0
	9	- .5		6	0
	T	- .5		8	0

Total Expectation = +2/30 = +6.67%

Bet Variation

The primary way to win money in blackjack is to recognize situations like this one, where basic strategy will show a profit, and bet more money accordingly. This exploitation of decks favorable for basic strategy will henceforth be referred to as gain from "bet variation." We will be concerned with the following questions: how often will the deck become rich, how rich can it get, and, in the next chapter, how can mortal players learn to diagnose this condition without working out the exact odds, as was done in our example.

Strategy Variation

Another potential source of profit is the recognition of when to deviate from the basic strategy. Keep in mind that, by definition, basic strategy is optimal for the full deck, but not necessarily for the many subdecks (like the previous five card example) encountered before reshuffling.

Basic strategy dictates hitting (5,8) v. 9, but in this particular situation the expectation for hitting is $-.50$, whereas standing would give a mathematical standoff since the dealer would bust half the time, as often as he had a six underneath. (If we survive our hit we only get a push, while a successful stand wins.) Similarly it's better to stand with (5,8) v T, (6,8) v 9, and (6,8) v T, for the same reason.

In each of the four cases we are 50% better off to violate the basic strategy, and if we had been aware of this we could have raised our basic strategy edge of 6.67% by another 6.67% to 13.33%. This extra gain occasionally available from appropriate departure from basic strategy, in response to fluctuations in deck composition which occur before reshuffling, will be attributed to "variations in strategy." Again, we will be concerned with how often it happens, how much it's worth, and, later, how we can exploit it.

As another exercise of this sort the ambitious reader should try to show that a six card residue consisting of [2, 4, 6, 7,9,T] has a -6.67% basic expectation, but a mammoth 27.22% expectation for precisely optimal play. Some of the departures from basic strategy are eye opening indeed and illustrate the wild fluctuations associated with extremely depleted decks. (Generally, variations in strategy can mitigate the disadvantage for compositions unfavorable for basic strategy, or make more profitable an already rich deck. This is a seldom encountered case in that variation in strategy swings the pendulum from unfavorable to favorable. Since these examples are exceedingly rare, the presumption that the only decks worth raising our bet on are those already favorable for basic strategy, although not entirely true, will be useful to maintain.)

Insurance is "linear"

A simple illustration of how quickly the variations can arise is the insurance bet. The basic strategist never insures, since his highest expectation (on the half unit wagered) is

$2 \times \left(\frac{16}{49}\right) - 1 \times \left(\frac{33}{49}\right) = -\frac{1}{49} = -2\%$, when neither of his cards is a ten. However, someone who plays two hands, and can look at both before insuring, might have four non-tens and thus recognize a $2 \times \left(\frac{16}{47}\right) - 1 \times \left(\frac{31}{47}\right) = +\frac{1}{47} = 2\%$ advantage.

Insurance is interesting for another reason; it is the one situation in blackjack which is truly "linear," being resolved by just one card (the dealer's hole card) rather than by a complex interaction of possibly several cards whose order of appearance could be vital. From the standpoint of settling the insurance bet, we might as well imagine that the value -1 has been painted on 35 cards in the deck and $+2$ daubed on the other 16 of them. The player's insurance expectation for any subdeck is then just the sum of these "payoffs" divided by the number of cards left. This leads to an extraordinarily simple mathematical solution to any questions about how much money can be made from the insurance bet (if every player in Nevada made perfect insurance bets it might cost the casinos about 40 million dollars a year), but unfortunately other manifestations of the spectrum of opportunity are not so uncomplicatedly linear.

Approximating Bet Variation

The task, when confronted with something more complex like variations in betting, is to grope for some approximate representation, conceptually as simple as the insurance structure, in order to avoid the impossibly lengthy computer calculations which would be necessary to analyze all possible subsets of a blackjack deck. Ultimate justification of the following method will be deferred to the Appendix for the mathematically inclined.[A,B,C]

Suppose we desired to approximate basic strategy favorability for any subset of cards which might be encountered before reshuffling, and we wished to do this by again painting "payoffs" (analogous to the +2 and —1 values appropriate for insurance) on each of the 52 cards. Our problem is to select these 52 numbers (which will replace, for our immediate purposes, the original denominations of the cards) so that the average value of the remaining payoffs will be very nearly equal to the true basic strategy expectation for any particular subset.

Using a traditional mathematical measurement of the accuracy of our approximation called the "method of least squares," it can be shown that the appropriate numbers are, as intuition would suggest, the same for all cards of the same denomination:

Best Linear Estimates of Deck Favorability (in %)

A	2	3	4	5	6	7	8	9	T
31.1	-19.4	-22.4	-28.0	-35.2	-23.4	-14.3	0.0	9.2	26.0

The numbers are derived by multiplying the effect of removing a single card from the full 52 card pack (on the player's basic strategy expectation) by —51. To assert that these are "best" estimates under the criterion of least squares means that, although another choice might work better in occasional situations, this selection is *guaranteed* to minimize the overall average squared discrepancy between the true expectation and our estimate of it.

How do we use them? Suppose we're considering the residue [2,4,6,7,9,T] mentioned earlier and we want to estimate the basic strategy expectation (which we already know to be -6.67%). We add the six payoffs corresponding to these cards $-19.4 -28.0 -23.4 -14.3 + 9.2 + 26.0 = -49.9$ and then divide by six, to average: $-\frac{49.9}{6} = -8.32$ (in %). It is the ensemble of squared differences between numbers like -6.67%, the true expectation, and -8.32%, our estimate of it, which least squares minimizes.

The estimate is not astoundingly good in this small subset case, but accuracy is much better for larger subsets, necessarily becoming perfect for 51 card decks. A subsequent simulation study mentioned in Chapter Four indicates the technique is quite satisfactory in the first 2/3 of the deck, where it is of most practical interest, considering casino shuffling practices.

Approximating Strategy Variation

The player's many different possible variations in strategy can be thought of as many embedded subgames, and they too are amenable to this sort of linearizing. Precisely which choices of strategy may confront the player will not be known, of course, until the hand is dealt, and this is in contrast to the betting decision which is made before every hand.

Consider the player who holds a total of 16 when the dealer shows a ten. The exact cards the player's total comprises are important only as they reveal information about the remaining cards in the deck, so suppose temporarily that the player possesses a piece of paper on which is written his current total of 16, and that the game of "16 versus Ten" is played from a 51 card deck. (52 less the dealer's ten.) Computer calculations show that the player who draws a card to such an abstract total of 16 has an expectation of $-.528$, while if he stands on 16 his expectation is $-.535$. He is therefore .007 better off to draw than stand for a full 51 card deck.

Suppose now that it is known that one five has been removed from the deck. Faced with this reduced 50 card deck the player's expectation by drawing is −.559 while his expectation by not drawing is −.540. In this case, he should stand on 16; the effect of the removal of one five is a reduction of the original .007 favorability for drawing by .026 to −.019. In similar fashion one can determine the effect of the removal of each type of card. These effects are given below, where for convenience of display we switch from decimals to per cent.

Effects of Removal on Favorability of hitting 16 v. Ten

Card Removed	A	2	3	4	5	6	7	8	9	T
Effect (%)	-.5	-.3	-.8	-1.7	-2.6	1.7	-.7	0	.6	1.2

The average of these effects for the 51 card deck is zero since the player's gain in expectation by hitting is unchanged over all such removals.

Now we construct a one card payoff game of the type already mentioned, where the player's payoff is given by

$$P_i = .7 \quad -50E_i .$$

E_i is the effect, just described, of the removal of the i^{th} card. Approximate determination of whether the blackjack player should hit or stand for a particular subset of the deck can be made by averaging these payoffs. Their average value for any subset is our "best linear estimate" of how much (in %) would be gained or lost by hitting.

Similarly, any of the several hundred playing decisions can be approximated by assigning appropriate single card payoffs to the distinct denominations of the blackjack deck. The distribution of favorability for changing (violating) basic strategy can be studied further by using the well known normal distribution of traditional statistics to determine how often the situations arise and how much can be gained when they do. Derivation of this method is also reserved for the Appendix. [D,E]

How much can be gained by Perfect Play?

An abbreviated guide to the spectrum of opportunity for perfect play appears in the following table which relates the number of unseen cards to the amount (in %) which can be obtained at that point from insurance, other variations in strategy exclusive of insurance, and betting one extra unit when the deck is favorable for basic strategy.

Number of Unseen Cards	Insurance Gain	Strategy Gain (no Insurance)	Betting Gain
10	.51	4.22	2.65
15	.36	2.68	2.08
20	.27	1.81	1.65
25	.21	1.23	1.39
30	.15	.83	1.13
35	.11	.53	.93
40	.07	.31	.71
45	.03	.15	.52

The somewhat complicated formula which governs these fluctuations verifies what we can see from the table, namely the dependence of the amount gained on the depth to which the deck is dealt. This is consistent, of course, with Thorp and Walden's 'Fundamental Theorem'. Two other important determiners of how much can be gained from individual strategy variations are also pinpointed by the formula.

Average Disadvantage for Violating Basic Strategy

In general, the greater the loss from violating the basic strategy for a full deck, the less frequent will be the opportunity for a particular strategy change. For example, failure to double down 11 v 3 would cost the player 29% with a full deck, while hitting a total of 13 against the same card would carry only a 4% penalty. Hence, the latter change in strategy can be expected to arise much more quickly than the former, sometimes as soon as the second round of play.

Volatility

Some plays are quite unfavorable for a full deck, but nevertheless possess a great "volatility" which will overcome the previous factor. Consider the effects of removal on, and full deck gain from, hitting 14 against a four and also against a nine:

	Effects of Removal for Hitting 14										Full Deck Gain by Hitting
	A	2	3	4	5	6	7	8	9	T	
v. 4	-.8	-.5	-2.0	-2.5	-2.9	-3.0	-2.9	2.0	2.3	2.6	-14.3
v. 9	-.2	0	.2	-.2	-1.1	-1.8	-1.9	.1	.6	1.1	10.6

Despite the fact that we're worse off changing strategy in the full deck by 14.3% to 10.6%, proper knowledge of hitting 14 v 4 is worth more than three times as much as knowing when to stand with 14 v 9 as we will see in the next table.

This is because large effects of removal are characteristic of hitting stiff hands against small cards and hence these plays can become quite valuable deep in the deck despite being very unfavorable initially. This is not true of the option of standing with stiffs against big cards, which plays tend to be associated with small effects. In the first case an abundance of small cards favors both the player's hitting and the dealer's hand, doubly increasing the motivation to hit the stiff against a small card which the dealer is unlikely to break. In the second case an abundance of high cards is unfavorable for the player's hitting, but is favorable for the dealer's hand; these contradictory effects tend to mute the gain achievable by standing with stiffs against big cards.

We can liken the full deck loss from violating basic strategy to the distance that has to be traveled before the threshold of strategy change is reached. The effects of removal (or more precisely their squares, as we shall learn) are the forces which can produce the necessary motion.

The following table breaks down strategy variation into each separate component and was prepared by the normal approximation methods. The averaging assumes a penetration of the deck such that variations in strategy are equally likely to be contemplated from n = 10 to n = 49 cards remaining. This should roughly approximate dealing three quarters of the deck, shuffling up with 13 or fewer cards remaining before the start of a hand, but otherwise finishing a hand in progress.[F]

AVERAGE GAINS FOR VARYING BASIC STRATEGY
(Thousandth of a percent)

Player's Hard Total \ Dealer's Card	2	3	4	5	6	7	8	9	T	A
17						1	11	4	6	9
16	6	6	5	4	8	20	15	19	95	7
15	12	10	8	7	8	9	8	10	47	6
14	19	16	12	9	10	6	3	4	50	3
13	34	28	20	14	17	2	1	4	36	2
12	23	29	30	22	28			5	20	1
11	1	1	1	1	1	4	6	7	22	13
10	1	1	1	1	1	3	4	6	18	8
9	8	6	4	2	2	10				
8	1	2	5	8	7	1				
7			1	2	3					
Hitting Soft 18							1	1	4	2
Soft Double										
A9	1	2	2	3	3					
A8	2	3	4	5	4					
A7	4	4	3	2	1	1				
A6	3	3	1	1		1				
A5	2	3	2	1						
A4	1	2	2	1	1					
A3	1	1	2	2	1					
A2	1	1	3	2	2					
Pairs										
TT	9	14	18	25	27					
99	1	1	1	1	1					
88									1	
66	1	1				1				
33						1				
22		1	1			1				
AA									1	1
Insurance										186

Gains from not doubling 10 and 11 against small cards have really been undervalued in the table since we have neglected the opportunities opened up to hit the subsequently developed stiff hands. Gain from the latter activity is (perhaps unfairly) recorded in the 12-17 rows.

Similarly the methodology incorrectly assesses situations where drawing only one card is dominated by a standing strategy, but drawing more than once is preferable to both. An example of this could arise when the player has 13 against the dealer's ten and the remaining six cards consist of four 4's and two tens. The expectation by drawing only one card is −.6000, by standing −.4667, but by drawing twice it is −.2000. The next higher step of approximation, an interactive model of blackjack, would pick this sort of thing up, but it's doubtful that the minuscule increase in accuracy would be balanced by the difficulty of developing and applying the theory.

Remember, the opportunities we have been discussing will be there whether we perceive them or not. When we consider the problem of programming the human mind to play blackjack we must abandon the idea of determining instantaneous strategy by the exhaustive algorithm described in the earlier parts of the book. The best we can reasonably expect is that the player be trained to react to the proportions of different denominations remaining in the deck. Clearly, the information available to mortal card counters will be imperfect; how it can be best obtained and processed for actual play will be the subject of our next chapter.

APPENDIX TO CHAPTER 3

A.

To build an approximation to what goes on in an arbitrary subset, let's assume a model in which the favorability of hitting 16 vs Ten is regarded as a linear function of the cards remaining in the deck at any instant. For specificity let there remain exactly 20 cards in the deck.

Model: $$Y_i = \sum_{j=1}^{51} \beta_j X_{ij}$$

Y_i = Favorability of hitting for i^{th} subset of 20 cards.

X_{ij} = 0 or 1 reflecting absence or presence of j^{th} card in i^{th} subset

Normal Equations:
Least square estimates satisfy $X'X\beta = X'Y$

Dimensions:

X is $\binom{51}{20} \times 51$ $\qquad$ β is only 51×1 $\qquad$ Y is $\binom{51}{20} \times 1$

What would be the best choice for these linear weights? Gauss's principle of least squares estimation leads us to the normal equations $X'X\beta = X'Y$ for fitting the best hyperplane to $\binom{51}{20} \sim 78$ trillion points in 52 space.

Y is the vector of favorabilities associated with each subset of the full deck, X is a matrix each of whose rows contains 20 1's and 31 0's, and the solution, β, will provide us with our 51x1 vector of desired coefficients.

All we must do is:

a. Run the computer day and night to determine the Y's.

b. Premultiply a $\binom{51}{20} \times 51$ matrix by its transpose.

c. Multiply the result of b. by β

d. Multiply X' by Y and finally

e. Solve the resultant system of 51 equations in 51 unknowns!

Fortunately many simplifications take place due to the special nature of $X'X$, whose diagonal elements are all $\binom{50}{19}$ and all of whose off diagonal elements are $\binom{49}{18}$, and we will be able to show the unique solutions are $\beta_j = \dfrac{\mu - 50E_j}{20}$, where

μ is the full deck gain of .7% for hitting 16 and the E_j are the effects of removal defined in the main part of the chapter.

For the more general derivation suppose there are k cards in the deck and least squares estimates of β_j $(j = 1,51)$ are sought for the model

$$Y_i = \sum_{j=1}^{51} \beta_j X_{ij} ,$$

where Y_i is the favorability of conducting basic strategy for the i^{th} subset and $X_{ij} = 0$ or 1 depending on the absence or presence of the j^{th} card in the subset.

The normal equations for the β_j will be

$$\sum_i X_{ij}^2 \beta_j + \sum_{m \neq j} \left\{ \sum_i X_{ij} X_{im} \right\} \beta_m = \sum_i Y_i X_{ij} \quad (j = 1,51)$$

If a complete sampling of all $\binom{51}{k}$ possible subsets is assumed, then

$$\sum_i X_{ij}^2 = \binom{50}{k-1} \quad \text{and} \quad \sum_i X_{ij} X_{im} = \binom{49}{k-2}$$

$\sum_i Y_i X_{ij}$ is the total favorability of all k card subsets containing the jth card. Since the average favorability of the $\binom{50}{k}$ subsets which do not contain the jth card is $\mu + E_j$ (by definition of E_j and a probabilistic argument which assumes k is large enough to guarantee resolution of the particular strategic situation without reshuffling), we have

$$\mu = \frac{\sum_i Y_i X_{ij} + \binom{50}{k} \cdot (\mu + E_j)}{\binom{51}{k}}$$

Solving and simplifying yields $\sum_i Y_i X_{ij} = \binom{50}{k-1} \cdot \mu - \binom{50}{k} E_j$.

The first of the normal equations becomes:

$$\binom{50}{k-1} \beta_1 + \binom{49}{k-2} \sum_{m=2}^{51} \beta_m = \binom{50}{k-1} \mu - \binom{50}{k} E_1$$

A permutation of subscripts changes all 51 equations to this form. The solutions for k = 50 are

$$\beta_j = \frac{\mu}{50} - E_j \text{ since } \sum_{m=2}^{51} \beta_m = \mu + E_1 .$$

Similarly, it can be shown by direct substitution that the solutions for any value of k are

$$\beta_j = \frac{\mu}{k} - \frac{50}{k} E_j = \frac{P_j}{k},$$ and the best (least squares) linear indicators for varying strategy are the single card payoffs, P_j. Their average value in a given subset is the corresponding estimate of favorability for carrying out the basic strategy.

Other aspects of blackjack, such as the player's expectation itself, or the drawing expectation, or the standing expectation separately, could be similarly treated. But, since basic strategy blackjack is so well understood it will minimize our error of approximation to use it as a base point, and only estimate the departures from it.

B.

Uniqueness of this solution follows from the non-singularity of a matrix of the form $\begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{pmatrix}$, with a throughout the main diagonal and $b \neq a$ in every non-diagonal position. The proof is most easily given by induction.

Let D(n,a,b) be the determinant of such an n x n matrix. We shall prove $D(n,a,b) = (a-b)^{n-1}(a+(n-1)b)$. This is true for n=1 obviously. We have $D(n-1,a,b) =$

$a \cdot D\left(n-1, \frac{a^2-b^2}{a} \quad \frac{ab-b^2}{a}\right)$ by multiplying the first row of our original matrix by b/a and subtracting it from the last n—1 rows to produce $\begin{pmatrix} a & b & b & b \\ 0 & c & d & d \\ 0 & d & c & d \\ 0 & d & d & c \end{pmatrix}$, where

$c = \dfrac{a^2-b^2}{a}$ and $d = \dfrac{ab-b^2}{a}$. Applying the inductive hypothesis that the theorem is true for dimension n—1 we have $D(n,a,b) =$

$$\frac{a}{a^{n-1}} D\left(n-1, a^2-b^2, ab-b^2\right) = \frac{(a-b)^{n-1}}{a^{n-2}} D(n-1, a+b, b) =$$

$$\frac{(a-b)^{n-1}}{a^{n-2}} \cdot a^{n-2} \cdot \left(a+b+(n-2)\cdot b\right) = (a-b)^{n-1}\left(a+(n-1)\, b\right),$$

as was to be shown.

C.

This derivation has much the flavor of a typical regression problem, but in truth it is not quite of that genre. Y_i is the true conditional mean for a specified set of our regression variables X_{ij}. It would be wonderful indeed if Y_i were truly the linear conditional mean hypothesized in regression theory, for then our estimation techniques would be perfect. But here we appeal to the method of least squares not to estimate what is assumed to be linear, but rather to best approximate what is almost certainly not quite so.

If we define a random variable Y to be the actual amount gained or lost by hitting on a particular play from the ith subdeck then Y has possible values $\{2,1,0,-1,-2\}$ with $EY = Y_i$. This emphasizes that Y_i is a fixed number we are trying to *approximate* as a linear function of the X_{ij}, and not a particular observation of a random variable as it would be in most least squares fits.

D.

Suppose σ^2 is the variance of the single card payoffs and μ is their full deck average value. Let

$$b^2 = \mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}\left(\frac{52-n}{51}\right)$$ be the variance for the mean of a sample of size n drawn without replacement from a finite pack of cards.

By a version of the Central Limit Theorem for correlated summands, $\bar{x}$ is approximately normally distributed. Assume $\mu \leq 0$ and that the card counter only changes strategy or bet when it is favorable to do so. (Assuming $\mu \leq 0$ is equivalent to redefining the single card payoffs, if necessary, so they best estimate the favorability of altering the basic strategy.) The card counter's expectation with n cards remaining, E(n), can be approximated by an integral:

$$E(n) \sim \frac{1}{\sqrt{2\pi}\, b} \int_0^{\infty} \bar{x}\, e^{-\frac{1}{2}\left(\frac{\bar{x}-\mu}{b}\right)^2} d\bar{x}$$

Standardizing $\bar{x}$ by $y = \frac{\bar{x}-\mu}{b}$ we have

$$E(n) \sim \frac{b}{\sqrt{2\pi}} e^{\frac{-\mu^2}{2b^2}} - |\mu| \cdot \int_{\frac{|\mu|}{b}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

(The integral in the second term is an approximation to the probability that the bet will be favorable.)

The Central Limit Theorem appealed to appears in the exercises of Fisz. It states that if $n \leq \frac{N}{2}$ and $n, N \longrightarrow \infty$ where n is the sample size and N the population size, and for every $\epsilon > o$, limit $\frac{1}{\sigma^2 N} \cdot \sum_{|X_j-\mu| \geq \epsilon n \sigma_{\bar{x}}} (X_j - \mu)^2 = 0$,

then $$\frac{\bar{x}-\mu}{\sigma_{\bar{x}}} \longrightarrow N(0,1)$$ in distribution.

The applicability is easily verified in our case since the X_i are our "payoffs" and are all bounded. For sufficiently large N and n there will be no terms in the sum since

$$\epsilon\, n\sigma_{\overline{X}} = \epsilon \sqrt{\frac{N-n}{N-1}}\,\sqrt{n}\,\sigma > \frac{\sqrt{n}\,\epsilon\sigma}{\sqrt{2}} \longrightarrow \infty$$

This proves the convergence for $n \leq \frac{N}{2}$, but also establishes the theorem for samples of size N—n by virtue of

the fact that $\frac{N}{n}\mu = \overline{X}_n + \left(\frac{N}{n} - 1\right)\overline{X}_{N-n}$ and we can let

$n, N \longrightarrow \infty$ with $\frac{N}{n}$ held constant.

Of course in practical application N is finite (rarely exceeding 312) and the proof of the pudding is in the eating.

Another representation of E(n), which will prove more convenient for tabulation, is in terms of what is sometimes called the "Unit Normal Linear Loss Integral":

$$E(n) \sim b \int_{\frac{|\mu|}{b}}^{\infty} \left(z - \frac{|\mu|}{b}\right) N(z)\,dz$$

where N(z) is the standard normal density.

Differentiation of this relation by each variable separately shows that E(n) is an increasing function of b and a decreasing function of $|\mu|$. Since $b = \sigma\sqrt{\frac{N}{n(N-1)} - \frac{1}{N-1}}$ we have that E(n) increases with diminishing n and with increasing σ. The former relation is consistent with Thorp and Walden's Fundamental Theorem and the latter pinpoints the volatility parameter.

E.

As a specific example of the accuracy of the normal approximation we will look at the insurance bet. Because of the probabilistic anomaly that the insurance expectation available with $3k+1$ cards remaining is the same as for $3k$ cards, it is appropriate to compare the continuous method over a span of three consecutive numbers of unplayed cards. Figures are presented in the table for 20, 21, and 22 cards remaining.

GAIN FROM INSURANCE

Cards Remaining	Actual Gain	Normal Approximation
20	.002796	.002741
21	.002499	.002597
22	.002499	.002459
Total	.007794	.007797
Average	.002598	.002599

We see that, in situations like insurance, the normal approximation might even be more desirable than the exact value. The smoothing of the continuous method irons out the discreteness and provides perhaps a more representative answer to a question like "What happens at about the twenty card level?"

F.

The Table of Average Gains for Varying Basic Strategy was constructed by establishing the frequency of dealer up cards as 1/13 for twos through nines, 35/663 for playable aces, and 188/663 for playable tens. Out of every 1326 player hands, it was assumed he would face a decision with totals of hard seven through eleven 32, 38, 48, 54, and 64 times respectively, with each soft double 16 times, and with a soft 18-hit on 23 occasions. Different frequencies were used for the hard totals of twelve through seventeen, depending on the dealer's up card. For small cards (2-6) these were estimated to be 130, 130, 110, 110, 100, and 100 respectively, while for high cards the figures 150, 155, 160, 165, 165 and 180 were used. Obviously some dependence is neglected, such as that between the player's hand and dealer's card as well as that if, for example, we make a non-basic stand with fourteen it reduces the frequency of fifteens and sixteens we might stand with.

4

EXPLOITING THE SPECTRUM — SINGLE PARAMETER CARD COUNTING SYSTEMS

"You count sixteen tens and what do you get?
Another day older and deeper in debt."
—Anonymous card counter's lament—

In their Fundamental Theorem paper, Thorp and Walden provided the simplest possible illustration of the spectrum of opportunity. Suppose a standard deck of cards is dealt through one at a time, without reshuffling. Before each card is turned the player has the option of wagering, at even money, that the next card will be red. For a full deck the game has a zero expectation, but after the first card is played the deck will be favorable for the wager on red about half the time.

An optimal card counting strategy is obvious, so for a more interesting illustration we'll assume the player is color blind. One can imagine several methods which will show a profit but fall short of optimality.

One idea is to look for an excess of hearts over spades among the unplayed cards. When this condition obtains, the player should on the average, but not always, have the advantage. We'll call this system A.

The diamond counter might employ a system B, monitoring the proportion of diamonds in the deck and betting on red when diamonds constitute more than one-fourth of the remaining cards. Yet a third possibility would be system C, based on the relative balance between three suits, say clubs, hearts, and diamonds. Since on the average there are twice as many red cards as clubs, the deck should tend to be favorable whenever the remaining red cards are more than twice as numerous as the clubs.

All three of these card counting methods can be carried out by assigning point values to the cards remaining in the deck, which point values would be opposite in algebraic sign to the numbers counted and continuously added as the cards are removed from the deck. The appropriate point values for the three systems discussed, as well as the payoffs for the game itself, are given below.

System	Spade	Heart	Diamond	Club	Sum of Squares	Correlation
A	-1	1	0	0	2	.707
B	-1	-1	3	-1	12	.577
C	0	1	1	-2	6	.816
Payoffs	-1	1	1	-1	4	1.000

In my search for an explanation of how it was that different card counting systems would be able to interpret and exploit a blackjack deck, I decided to exhaustively analyze this simplest of all possible games. To my way of thinking the example had two advantages. First of all, I could program the computer to determine precisely how much could be gained at any deck level with the three systems, as well as with the optimal color dependent strategy given by the payoffs themselves—there would be no sampling error since exact probabilities would be used. The second advantage was that the very simplicity in structure might make evident the direction to pursue in analyzing the manifoldly more complex game of blackjack.

The Role of the Correlation Coefficient

The results of this program, run on June 17, 1974 at 1824, and taking 2.94 seconds to execute, were to me what the unchanging speed of light through the ether must have been to Michelson and Morley. For some reason, which I can no longer recollect, I had already calculated what, in statistics, is called the correlation coefficient between the point values of the card counting systems and the payoff for the game itself. This is done by dividing the sum of the products of the respective values assigned to each suit and the payoffs for the suit by the square root of the product of the sum of squares of values for the card counting system and sum of squares of the payoffs.

As an example, for system C our correlation coefficient is obtainable from the following arithmetic

Cards	System C	Payoffs	C^2	P^2	$C \cdot P$
Spades	0	-1	0	1	0
Hearts	1	1	1	1	1
Diamonds	1	1	1	1	1
Clubs	-2	-1	4	1	2
			6	4	4

$$\text{Correlation} = \frac{\sum C_i P_i}{\sqrt{\sum C_i^2 \cdot \sum P_i^2}} = \frac{4}{\sqrt{6 \cdot 4}} = \frac{4}{\sqrt{24}} = .816$$

Similarly for B we get a correlation of $\frac{4}{\sqrt{12 \cdot 4}} = .577$ and for

A, $\frac{2}{\sqrt{2 \cdot 4}} = .707$

What leaps out of the following abbreviated table of results is the fact that the relative amount gained by each of the card counting systems tends to cluster near the system's correlation, regardless of the number of cards left in the deck.

Relative Amount of Total Profit Gained by Red-Black Systems

Number of Cards Left	A	B	C
9	.676	.569	.768
18	.718	.607	.833
27	.691	.574	.804
36	.719	.573	.842
45	.669	.569	.764

With this hint, it was not difficult to extend to a higher dimension the normal distribution analysis mentioned in

Chapter Three and derive a "bivariate" relation between what I defined as the efficiency of a card counting system and its correlation with the particular situation at hand. This derivation appears in the Chapter Appendix. [A,B,C,D]

Efficiency

In mechanics the term "efficiency" is used as the ratio of the actual to the ideal, the quotient of work done by a machine and work put in. With this in mind it seems natural to define the efficiency of a card counting system to be the ratio of the profit accruing from using the system to the total gain possible from perfect knowledge and interpretation of the unplayed set of cards. What we learn from the mathematics, then, is that efficiency is directly related, and in some cases equal, to the correlation between the point values of the card counting system and the single card payoffs approximating the blackjack situation considered.

In blackjack we have one card counting system which may be used for a variety of purposes; first of all to determine if the deck is favorable to the player or not, and secondly to conduct any of more than a hundred different variations in strategy which might arise after the hand is dealt. We can consider the card counting system to be an assignment of point values to the cards remaining in or deleted from the deck, at our convenience.

In theory any assignment of points is permissible, but simple integers are more tractable for the human memory. In addition it is desirable to have the restriction that the count be balanced in that the sum of the point values for a full deck be zero. This way the direction of deflection of the deck from normal is instantaneously evident from the algebraic sign of the running count, regardless of depth in the deck.

Betting Correlation

As a first example of the efficiency of a blackjack system, we will look at the most frequent and important decision, namely whether to bet extra money on the hand about to be dealt. The index of this capability to diagnose favorable decks will be the correlation coefficient between the point values of the count system and the best linear estimates of deck favorability mentioned in the previous chapter.

It is a mathematical fact that "correlation is invariant under linear transformation," and this justifies the arithmetic simplification of correlating with the effects of removing a single card, rather than the single card payoffs derived from these effects. To calculate, for instance, the betting strength of the HiOpt II system, we have

	A	2	3	4	5	6	7	8	9	T	Sum of Squares
Hi Opt II Point Values	0	1	1	2	2	1	1	0	0	-2	28
Effects of Removal	-.61	.38	.44	.55	.69	.46	.28	-.00	-.18	-.51	2.84

The sum of products (or "inner product" as mathematicians express it) will be

$$1\text{x}(.38+.44+.46+.28)+2\text{x}(.55+.69)-2\text{x}(-.51)\text{x}4 = 8.12$$

(Remember there are really four ten valued denominations but we condense the table to avoid repetition.) The correlation for betting purposes is $\dfrac{8.12}{\sqrt{28(2.84)}} = .91$

To illustrate how betting effectiveness is related to this correlation, I will quote from Gwynn's simulation study which involved basic strategy play of two sets of twenty million hands, one under the Las Vegas Strip rules generally presumed in this paper and the other according to Reno rules, where the dealer hits soft 17 and the player's double down option is restricted to totals of ten and eleven. A single deck was dealt down to the 14 card level and all systems were evaluated according to their simultaneous diagnosis of the same hands, so the element of luck was kept to a minimum. If the pre-deal count for a system suggested an advantage, one unit was bet, otherwise nothing.

The systems are described in the table by their ten point values, ace through ten from left to right. This is followed by the betting correlation, the Las Vegas simulation yield in % of units gained per hand played, the predicted yield (in parenthesis), the Reno simulation yield, and again the predicted yield using the methods of this section. Slightly altered parameters are necessary to approximate the Reno game, but these correlations are not presented since they differ very little from those for Las Vegas.

System										Correlation	Las Vegas Gain	Reno Gain
-1	1	1	1	1	1	0	0	0	-1	.97	.82 (.85)	.64 (.66)
0	0	1	1	1	1	0	0	0	-1	.88	.74 (.77)	.56 (.59)
0	1	1	1	1	1	0	0	-1	-1	.89	.73 (.78)	.58 (.61)
-1	1	1	1	1	1	1	0	-1	-1	.96	.82 (.85)	.64 (.66)
0	1	1	1	1	0	0	0	0	-1	.86	.73 (.75)	.56 (.58)
-1	0	1	1	1	1	1	0	0	-1	.95	.82 (.84)	.63 (.64)
-1	0	0	0	1	0	0	0	0	0	.54	.45 (.46)	.30 (.28)
-2	1	2	2	2	2	1	0	0	-2	.98	.86 (.87)	.67 (.68)
0	1	1	2	2	1	1	0	0	-2	.91	.78 (.80)	.61 (.62)
0	1	2	2	3	2	2	1	-1	-3	.91	.78 (.80)	.61 (.61)
-2	1	2	2	3	2	1	0	-1	-2	.99	.87 (.87)	.67 (.68)
-4	2	3	3	4	3	2	0	-1	-3	1.00	.88 (.89)	.67 (.68)
0	2	2	3	4	2	1	0	-2	-3	.92	.77 (.79)	.61 (.63)
4	4	4	4	4	4	4	4	4	-9	.72	.64 (.63)	.44 (.45)
-9	5	6	8	11	6	4	0	-3	-7	1.00	.89 (.89)	.68 (.69)

Strategic Efficiency

The original, and still primary, interest in card counting systems has been in this sensitivity for detecting favorable decks. However, increased scrutiny by casino personnel makes wide variation in wagers impractical and there has evolved a secondary concern for how effective these systems would be for just varying strategy, particularly in single deck games. Reliable simulation estimates of this capability are extraordinarily time consuming, so the correlation method of analysis proves ideal for getting a fix on how much can be gained by these tactics.

With this in mind, a program was written to converge to the optimal point values for conducting the 70 variations of strategy associated with hard totals of 10 through 16, along with the insurance wager. The optimization was conducted with n=20 cards in the deck since this level was thought to be deep enough to be interesting but not unrealistically out of line with casino shuffling practices. Relative results for any card counting systems do not differ appreciably whether there are 10, 20, 30, or 40 cards remaining.

The initial optimization was to maximize overall strategic efficiency subject to a point value of −180 assigned to the tens. Subsequent optimization was conducted at different levels of complexity, level of complexity being defined as the maximum of the absolute values of the points assigned. Blackjack gurus seem unanimous in the opinion that the ace should be valued as zero since it behaves like a small card for strategic variations and a big card for betting strategy; this optimization also presumed such a value. The following table presents the champions of their respective divisions.

OPTIMAL SYSTEMS FOR VARIATION OF STRATEGY

A	2	3	4	5	6	7	8	9	Ten	Efficiency
0	0	0	1	1	1	1	0	0	- 1	.637
0	1	1	2	2	2	1	0	- 1	- 2	.672
0	1	2	2	3	2	2	1	- 1	- 3	.690
0	1	2	3	4	3	3	1	- 1	- 4	.6913
0	2	2	4	5	4	3	1	- 1	- 5	.6909
0	67	93	132	177	131	122	46	-48	-180	.694
51	60	85	125	169	122	117	43	-52	-180	.703

A noteworthy observation is that, if the ace is to be counted zero, improvement in the second decimal cannot be achieved beyond level three. Also, bigger is not necessarily better; the level four system narrowly edges the level five system. For the evaluated systems all decisions are made on the basis of a single parameter, the average number of points remaining in the deck. Evidently the maximum efficiency possible for strategic variation with a single parameter system is of the order of 70% and one can come quite close to that without going beyond the third level.

Some other interesting evaluations follow. Considering the difficulty and likelihood of error in, for instance, trying to associate four points with five spots on the card and one point with seven spots on the card, it is extremely questionable whether a price tag of $200 for a less than optimal level four system is a bargain. Although the Ten Count, when parameterized as a point count, uses the numbers 4 and —9, it is certainly not at the 9th level of mental gymnastics—one keeps track of the proportion of tens by counting off the tens and non-tens as they leave the deck.

MISCELLANEOUS BLACKJACK SYSTEMS

A	2	3	4	5	6	7	8	9	T	Playing Efficiency
	1	1	1	1					-1	.574
		1	1	1	1				-1	.615
		1	1	1	1	1		-1	-1	.623
	1	1	1	1	1			-1	-1	.592
-1	1	1	1	1	1				-1	.510
-1		1	1	1	1	1			-1	.547
-1	1	1	1	1	1	1		-1	-1	.532
4	4	4	4	4	4	4	4	4	-9	.621
	1	1	1	1	1	1	1	1	-2	.617
	1	1	1	2	1	1	1		-2	.670
	1	1	2	2	1	1			-2	.671
	2	2	3	4	2	1		-2	-3	.657

Two other methods of evaluation, based on the mysterious infinite deck, included pair splitting and doubling down on any two cards as options. They gave very similar relative results for all the systems' strategy gains reported here, with only an occasional interchange of the order of two systems whose efficiencies differ only in the third digit after the decimal point.

Proper Balance between Betting and Playing Strength

The proper relative importance to attach to betting efficiency and playing efficiency depends on several factors: depth of penetration, permissible increase in bet, and playing efficiency restricted to favorable decks. Assuming the same penetration used in the previously mentioned Gwynn simulations the following empirical formula provides such a weighting by estimating the average profit available in terms of the basic betting unit. If K units are bet on all decks diagnosed as favorable and one unit is bet otherwise, the average improvement due to card counting is approximately

$$[8(K-1) \cdot BE + 5(K+1) \cdot PE]/1000$$

units per hand, where BE is betting efficiency and PE is playing efficiency. (One should allow about 20% more for Las Vegas rules and 10% less for Reno.) The formula suggests the two efficiencies are almost equally important for a 1 to 4 betting scale and that betting efficiency is rarely more than one and a half times as important as playing efficiency.

In summary, then, the player who is shopping around for a best single parameter card counting system has a choice between

	Strategy Efficiency	Betting Efficiency
Best Strategy System	70%	90%
Best Betting System	55%	100%

Simplicity versus Complexity

"It is my experience that it is rather more difficult to recapture directness and simplicity than to advance in the direction of evermore sophistication and complexity. Any third-rate engineer or researcher can increase complexity; but it takes a flair of real insight to make things simple again."

E.F. Schumacher, **Small is Beautiful**

Now, when you build a better mousetrap the world will beat a path to your door, just as a reputation as a blackjack "expert" entitles one to crank letters on the topic "what system should I use?" Consider the following excerpt: "I go to Vegas every two weeks. . .I almost always come home with $2,000 or more betting only $5 chips. . .I would like the indices to be perfect. . .I shouldn't be looking for a better strategy since I do so well but I want to use the best."

Obviously I can do little to help this gentleman, but for others who have not been quite so successful I would advise that they avoid the awkward integers associated with the more complex counts. Using the best one-level system you can achieve either a 64% playing efficiency or a 97% betting efficiency, and so the small sacrifice seems justified when we consider the ease on the memory as well as the decreased likelihood of error.

In addition, the simple plus or minus one systems are much more easily modified by inclusion of other information. We will see in the next chapter that to raise strategy efficiency above 70% one must invoke separate parameters, and one of the easiest ways to do this is to use knowledge of the uncounted (zero-valued) cards not recognized by a simple level one system.

APPENDIX TO CHAPTER 4

A.

To derive the approximate relation between a card counting system and a single card payoff game (with a fixed number of cards remaining in the deck) we will assume that X, the average payoff for the n card subset, and Y, the average point value for the card counting system, are bivariate normal with correlation coefficient ρ and marginal distribution for X as in the appendix to Chapter Three. For simplicity assume a balanced point count normed so that Y is N(0,1). We have the density

$$f(x,y) = \frac{1}{2\pi(1-\rho^2)\sigma_X} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)y + y^2\right)\right]$$

and the relation

$$E(X \mid Y = y) = \mu_X + \rho\frac{\sigma_X}{\sigma_y}(y-\mu_y) = \mu_X + \rho\sigma_X y$$

Setting the conditional expectation equal to zero and assuming again that $\mu_X \leq 0$ we find the critical index for the card counting system to accept the wager to be

$$y > \frac{|\mu_X|}{\rho\sigma_X}$$

(We may assume $\rho > 0$, since, if not, the action will be presumed whenever $y < \frac{\mu}{\rho\sigma_X}$ and the resultant formula will be the same with $|\rho|$ in place of ρ.)

Thus the gain available from using this system will be given by the double integral:

$$h(\rho) = \int_{-\infty}^{+\infty} \int_{\frac{|\mu_X|}{\rho\sigma_X}}^{+\infty} x \cdot f(x,y)\, dy\, dx$$

By completing the square on x in the exponent of the density and interchanging the order of integration we obtain:

$$h(\rho) = \frac{\rho\sigma_X}{\sqrt{2\pi}}\, e^{-\frac{\mu_X^2}{2\rho^2\sigma_X^2}} - \frac{|\mu_X|}{\sqrt{2\pi}} \int_{\frac{|\mu_X|}{\rho\sigma_X}}^{+\infty} e^{-y^2/2}\, dy$$

which is the same as E(n) from previous Appendix D, only with $\rho\sigma_X$ replacing b.

Defining the efficiency of a card counting system to be the ratio of profit from using the system to total profit possible, $\frac{h(\rho)}{E(n)}$, we see that if $\mu_X = 0$ the efficiency will be precisely equal to ρ, the correlation coefficient. In any case $h(\rho)$ can be shown to be an increasing function of ρ by the same argument which established E(n) to be an increasing function of σ.

Further, rewriting

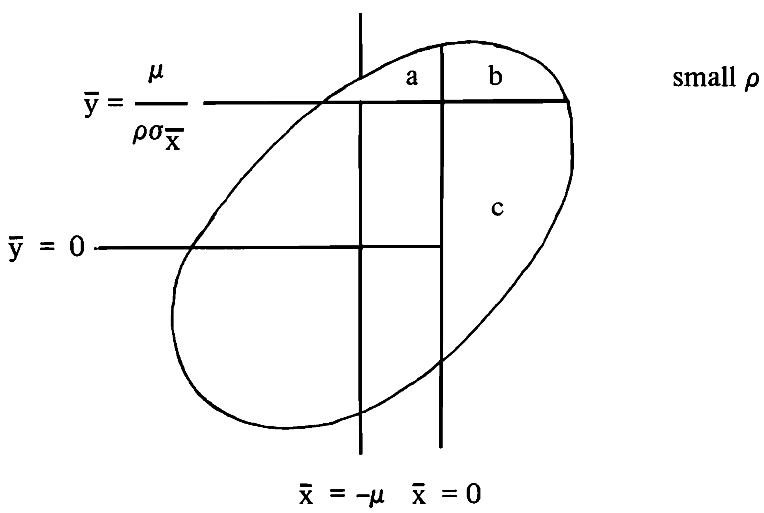
$$h(\rho) = \rho \left[\frac{\sigma_X e^{-\left(\frac{\mu_X}{\rho}\right)^2 / 2\sigma_X^2}}{\sqrt{2\pi}} - \left|\frac{\mu_X}{\rho}\right| \cdot \frac{1}{\sqrt{2\pi}} \int_{\left|\frac{\mu_X}{\rho}\right| / \sigma_X}^{+\infty} e^{-y^2/2}\, dy \right]$$

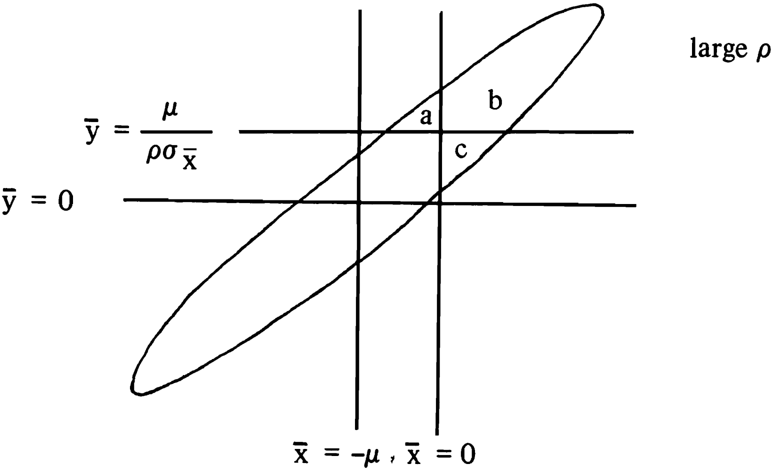
we may invoke the argument that E(n) decreases as $|\mu|$ increases to conclude, since $\left|\frac{\mu_X}{\rho}\right| \geq |\mu_X|$, that $h(\rho) \leq \rho \cdot E(n)$ and that efficiency will be less than ρ if $\rho < 1$ and $\mu_X \neq 0$. A similar but slightly more detailed discussion establishes that efficiency will improve as the deck is depleted, tending to approach the correlation coefficient ρ as a limit.

A look at the level contours of the associated bivariate normal distributions for small and large values of ρ is very enlightening. The average favorability for violating the basic strategy, $\overline{x}$, is marginally distributed with a mean of $-\mu$. The card count system presumes a favorable action when

$$\overline{y} > \frac{\mu}{\rho\sigma_{\overline{X}}}.$$

There are three regions of interest in the diagrams: (a) where action is taken but shouldn't have been, (b) where action is correctly taken, and (c) where action is not taken but should have been. The "ratio" of b—a to b + c corresponds to efficiency. (It is not really area we should compare here, but it aids understanding to view it that way.)





B.

As an example of this theory we can look at the insurance efficiencies of three card counting systems with n = 20, 21, and 22 cards left in the deck for comparison with the approximation of Chapter Three. The Einstein count of + 1 for 3,4,5, and 6 and —1 for tens results in a correlation of .85 for insurance. The Dubner Hi-Lo extends the Einstein values by counting the 2 as + 1 and the ace as −1, resulting in a correlation coefficient of .79. Another system, mentioned in **Beat the Dealer** by Thorp, extends Dubner's count by counting the 7 as + 1 and the 9 as −1, and has a correlation of .72 for insurance.

INSURANCE EFFICIENCIES

(a) actual and (b) approximate

	Einstein		Dubner Hi-Lo		Thorp Hi-Lo	
	.85		.79		.72	
n	(a)	(b)	(a)	(b)	(a)	(b)
20	.793	.806	.713	.723	.618	.630
21	.832	.804	.752	.720	.649	.626
22	.788	.801	.708	.718	.610	.622
Average	.804	.804	.724	.721	.626	.626

(The efficiencies are averaged with weighting by the gains)

Again, we see the "smoothing" advantages of the approximation.

C.

The assumption that evaluation of card counting systems in terms of their correlation coefficients for the 70 mentioned variations in strategy will be as successful as for the insurance bet is open to question. The insurance bet is, after all, a truly linear game, while the other variations in strategy involve more complex relations between several cards; these interactions are necessarily neglected by the bivariate normal methodology. There is one interesting comparison which can be made.

Epstein reports a simulation of seven million hands where variations in strategy were conducted by using the Ten-Count. An average expectation of 1.23% resulted when the deck was dealt down to, but not including, the last two cards. Averaging from n=3 to n=49 cards the gains presumed by the bivariate normal correlation method yields an improvement of 1.20% for the Ten Count above basic strategy, and a figure slightly over 2.00% for precisely optimal play. In today's casino conditions the deck will rarely be dealt this deeply, and half the previous figures would be more realistic.

D.

It might also be mentioned that correlation is undisturbed by the sampling without replacement. To prove this, let X_i be the payoff associated with the ith card in the deck and Y_i be the point value associated with the ith card in the deck by some card counting system. Since $\sum_{j=1}^{52} Y_j = 0$, we have

$$0 = \sum_{i=1}^{52} X_i \cdot \sum_{j=1}^{52} Y_i = \sum_{i=1}^{52} X_i Y_i + \sum_{i \neq j} X_i Y_j .$$ Hence

$$\underset{i \neq j}{E(X_i Y_j)} = \frac{\sum_{i \neq j} X_i Y_j}{52 \cdot 51} = -\frac{\sum_{i=1}^{52} X_i Y_i}{52 \cdot 51} = -\frac{E(XY)}{51} .$$ We

seek the correlation of $\sum^{n} X$ and $\sum^{n} Y$ for n card subsets.

This will be

$$\frac{E\left(\sum^{n} X \cdot \sum^{n} Y\right)}{\sqrt{\operatorname{Var}\left(\sum^{n} X\right) \cdot \operatorname{Var}\left(\sum^{n} Y\right)}} = \frac{n\,E(XY) + n(n-1)\,\underset{i \neq j}{E\left(X_i Y_j\right)}}{n\left(\frac{52-n}{51}\right) \sigma_x \quad \sigma_y}$$

$$= \frac{E(XY)\left[1 + \frac{1-n}{51}\right]}{\frac{52-n}{51} \quad \sigma_x \quad \sigma_y} = \frac{E(XY)}{\sigma_x \; \sigma_y} = \rho,$$ the

correlation of the X and Y values in the entire pack of 52 cards.

5

MULTIPARAMETER CARD COUNTING SYSTEMS

"The attraction of the gamble was to show that they were men to whom \$10 or \$20 less or more at the end of a week was not a matter of great concern."
Evelyn Waugh, **The Loved One**

As was pointed out in the previous chapter, many experts prefer to assign a value of zero to the ace in order to achieve a higher playing efficiency. They then recommend keeping a separate, or "side," count of the aces in order to adjust their primary count for betting purposes. Let's take a look at how this is done and what the likely effect will be.

Consider the Hi Opt I, or Einstein, count, which has a betting correlation of .88.

	A	2	3	4	5	6	7	8	9	T
Hi Opt I	0	0	1	1	1	1	0	0	0	-1
Betting Effect	-.61	.38	.44	.55	.69	.46	.28	.00	-.18	-.51

The average absolute effect of the Einstein monitored cards is $\frac{.44 + .55 + .69 + .46 + 4 \times (.51)}{8} = .52$, just a bit less than that of the ace, the most important uncounted card. It therefore seems reasonable to regard an excess ace in the deck as meriting a temporary readjustment of the running count (for betting purposes only) by plus one point. Similarly, a deficient ace should produce a deduction (temporary, again) of one point.

As an example, suppose there are 39 cards remaining, a +1 count, but only one ace left. Should we regard the deck as favorable? Well, we're shy two aces since the expected distribution is three in 39 cards; therefore we deduct two points to give ourselves a temporary running count of −1 and regard the deck as probably disadvantageous. In like fashion, with a count of −1 but all four aces remaining in the last 26 cards we would presume an advantage on the basis of a +1 adjusted running count. It can be shown by the mathematics in the appendix that the net effect of this sort of activity will be to increase the system's betting correlation from .88 to .96.[A]

Keeping Track of a single Denomination

There are certain important situations in strategic variation which are not handled well by any of the single parameter card counting systems. Among these are knowing when to stand with 15 and 16 against a dealer 7 or 8 and knowing when to stand with 12, 13, and 14 against a dealer 9, Ten, or Ace. Before presenting a method to improve single parameter card counting systems it is useful to look at a quantification of the relative importance of the separate denominations of nontens in the deck. This quantification can be achieved by calculating the playing efficiency of a card count which assigns one point to each card except the denomination considered, which counts as −12.

These single denomination efficiencies with n = 20 cards left in the deck are as follows:

SINGLE DENOMINATION EFFICIENCIES

A	2	3	4	5	6	7	8	9
.016	.010	.030	.069	.128	.109	.118	.060	.048

The difficulty of incorporating the 7 and 8 in a point count (and to a certain extent the 6 and the 9) is that they occasionally behave as low cards and occasionally as high cards. The fixed sign of the point value obscures this and can only be overcome by assigning the value zero and keeping a separate track of the density of these zero valued cards for reference in appropriate situations.

The Importance of the Seven when You have Fourteen against a Ten

Again, consider the Einstein point values and assume a holding of 14 against the dealer's ten.

	A	2	3	4	5	6	7	8	9	T
Point Value	0	0	1	1	1	1	0	0	0	-1
Effect on hitting 14 v T (%)	-.08	.44	.17	-.26	-.77	-1.41	-4.21	.22	.77	1.28

The Einstein correlation for the effects of removal is a mediocre .49 and will produce very little gain for the play.

The average effect of removal for the eight cards recognized by the Einstein count is about 1% and this suggests that, if the deck is one seven short, that should be worth four Einstein points. The mathematically correct index for standing with 14 against a ten is an average point value above +.22 for the unplayed cards. Thus with a count of +4 and 39 cards remaining the index of +4/39 = +.10 dictates hitting. Suppose, however, that there was only one seven left in the deck. Since there are two less sevens than normal for 39 cards, we could adjust the count (temporarily, for this play only) to +12/39 = .31 and recognize a situation where standing is probably correct.

(It will save a lot of arguments to keep in mind that a change in strategy can be considered correct from three different perspectives which don't always coincide: it can be mathematically correct with respect to the actual deck composition confronted; it can be correct according to the deck composition a card counter's parameter entitles him to presume; and it can be correct depending on what actually happens at the table. I've seen many poor players insure a pair of tens when the dealer had a blackjack, but I've seen two and a half times more insure when the dealer didn't!)

Alternatively suppose a point count of +7/26 = .27, which indicates standing with 14 v. Ten. If there were three sevens left in the deck at this stage (one extra) we would adjust the point count downward to +3/26 = .12 and draw, trying for one of those sevens. Incorporation of the density of sevens raises our system's correlation from .49 to .97.[B]

Some of the multiparametric approaches to particular strategy changes are startling in their simplicity and power. We've already seen the importance of the seven for playing 14 v. Ten in conjunction with the Hi Opt I, or Einstein, count. Knowledge of the sevens alone, without the primary count at all, does almost as well: the rule "stand if the density of sevens is less than half of normal" is about 70% efficient, while recognized card counting systems seldom do better than 20%. The further simplification, "stand if there are no sevens," is almost as effective, being equivalent to the previous rule if less than half the deck remains.

For playing 16 v. Ten the remarkably elementary direction "stand when there are more sixes than fives remaining, hit otherwise," is more than 60% efficient. We will see in Chapter Eleven that it consistently out-performs both the Ten Count and Hi Opt I. Of course, these are highly specialized instructions, without broader applicability, and we should be in no haste to abandon our conventional methods in their favor.[B]

Ultimate human Capability

If one's ambition is to raise overall strategic efficiency beyond the 70% level, perhaps as high as 90%, it is imperative that the primary system be quite simple and hence allow great flexibility for incorporating several auxiliary, independent sources of information.

The ability to keep separate densities of aces, sevens, eights, and nines as well as the Einstein point count itself is not beyond a motivated and disciplined intellect. The memorization of strategy tables for the basic Einstein system as well as proper point values for the separated denominations in different strategic situations should be no problem for an individual who is so inclined. The increases in playing efficiency and betting correlation are exhibited below.

INCORPORATION OF ZERO VALUED CARDS INTO EINSTEIN SYSTEM

		Cards Incorporated				
	Basic System	A	A,7	A,7,8	A,7,8,9	A,7,8,9,2
Playing Efficiency	.615	.635	.736	.811	.870	.891
Betting Correlation	.88	.96	.97	.97	.97	.98

The ace is included first because of its importance for betting strategy. It is of little consequence strategically except for doubling down totals of eleven, particularly against a 7, 8, or 9, and totals of ten against a Ten or Ace.

Actually the compleat card counting fanatic who aspires to count separately five zero valued denominations is better off using the Gordon system which differs from Einstein's by counting the 2 rather than the 6. Although poorer initially than Einstein's system, it provides a better springboard for this level of ambition. The Gordon count, fortified with a proper valuation of aces, sixes, sevens, eights, and nines, scores .922 in playing efficiency and the same .98 in betting correlation. This may reasonably be supposed to define a possibly realizable upper bound to the ultimate capability of a human being playing an honest game of blackjack from a single deck.

The Effect of Grouping Cards

All of the previous discussion has been under the assumption that a *separate* track of each of the zero-valued cards is kept. David Heath suggested sometime ago a scheme of blocking the cards into three groups {2,3,4,5}, {6,7,8,9}, and {10,J, Q,K}. Using two measures, the differences between the first two groups and the tens, he then created a two dimensional strategy change graphic (resembling somewhat a guitar fingering chart).

Heath's system is equivalent to fortifying a primary Gordon count with information provided by the block of "middle" cards, {6,7,8,9}, there being no discrimination among these cards individually. As we can see from the following table of efficiencies for various blocks of cards properly used in support of the Gordon and Einstein systems, it would have been better to cut down on the number of cards in the blocked group.[C,D,E]

Primary Count	Auxiliary Grouping	Playing Efficiency
Gordon	{ 6,7,8,9 }	.740
Gordon	{ 6,7,8 }	.767
Gordon	{6,7 }	.741
Einstein	{ 7,8,9 }	.756
Einstein	{7,8 }	.761
Einstein	{7}	.722

The optimal strategy point values in Chapter Four show the 6, 7, and 8 function predominantly as low cards, with the 9 usually playing the role of a high card whose inclusion in the grouping {6,7,8,9} often cancels the effect of one of the others.

John Henry vs the Steam Engine

As a test, both of the theory espoused in this book and my own ability to use a multiparameter card counting system, I played and recorded 5000 hands. Each of them was analyzed by a computer to determine if basic strategy should have been changed and, if so, how much expectation could have been gained by such appropriate departure.

I, myself, made decisions as to whether I would have altered the conventional basic strategy, using my own version of the system accorded an efficiency of .870 on page 59. The following table displays how much expectation per hand I and the computer gained by our strategy changes. My gain (in%) appears first, followed by the computer's, the results for which are always at least as good as mine since it was the ultimate arbiter as to which decisions were correct and by how much.

Unseen Cards	Insurance Gain	Non-insurance Gain
8-12	.44/.46	3.11/3.66
13-17	.32/.34	1.46/2.07
18-22	.18/.19	1.46/1.76
23-27	.18/.19	.72/ .95
28-32	.05/.05	.54/ .69
33-37	.08/.08	.32/ .39
38-42	.05/.05	.20/ .24
43-47	.04/.04	.06/ .07

My overall efficiency, including insurance, was .819. The discrepancy between this and the theoretical .870 is traceable to my ignorance of precise parameters and errors in tracking the cards. This table should be compared with the one on page 28. [F]

The most productive hand was a double of hard seven v 5; with an ace, deuce, eight, nine and four tens remaining it was better than an undoubled draw by 70%. The most bizarre change was a double on hard 13 v 6; with three eights, two sixes, sevens, and tens, and one ace, two, three, and four, doubling was 61% better than standing, 18% better than merely drawing.

APPENDIX TO CHAPTER 5

A.

An indicator count, (−12 1 1 1 1 1 1 1 1 1), monitors the presence of aces in the deck and will be uncorrelated with the primary one if zero is the assigned point value. This is because the numerator of the correlation, the inner product between the primary and the ace indicator count, will be zero, merely being the sum of the point values of the primary system (assumed to be balanced). To the degree of validity of the bivariate normal approximation zero correlation is equivalent to independence.

Hence we are justified in taking the square root of the sum of squares of the original systems' correlations as the multiple correlation coefficient. For the situation discussed, we find the ace indicator count has a .38 correlation for betting purposes, so the multiple correlation coefficient will be

$$\sqrt{(.88)^2 + (.38)^2} = .96$$

B.

The seven indicator has a .84 correlation for the 14 v T play, which gives $\sqrt{(.84)^2 + (.49)^2} = .97$ for the multiple correlation when incorporating it with the Einstein count. The "Six-Five" system for playing 16 v. T has a correlation of .68, while the Ten Count's is .62 and the Hi Opt I's .64.

C.

We can use the theory of multiple correlation to derive a formula for the appropriate number of points to assign to a block of k zero-valued cards when using them to support a primary count system. However, since the assumption of linearity underlies this theory as well as the artifice of the single card payoffs, the demonstration can be more easily given from the latter vantage point, using only elementary algebra.

Let Y_i, i = 1 to 13, be the point values for our balanced primary count. It will be shown in the appendix to Chapter 7 that we are entitled to presume that the deflection of any denomination from its customary density of 1/13 should be proportional to the point value assigned to the denomination. Now, alter a 52 card deck by deleting Y_i cards of denomination i if $Y_i > 0$ and adding Y_j cards of denomination j for $Y_j < 0$. We still have 52 cards, but the point count of the deck is $\sum_{i=1}^{13} Y_i^2$. There has been a corresponding change in the total of the single card payoffs, $P_i = \mu - 51 E_i$, of

$$-\sum^{13} Y_i(\mu - 51E_i) = 51 \sum^{13} YE.$$

Thus, a count of plus one produces a change in this total of

$$\frac{51\sum^{13} YE}{\sum^{13} Y^2}$$

Returning to the full deck, remove one blocked card, the average payoff for which is

$$\frac{\sum^{k}(\mu - 51E)}{k} = \mu - \frac{51}{k}\sum^{k} E,$$

where the summing takes place over the blocked cards. Now, replace it with an "average" unblocked card whose payoff may be assumed to be

$$\frac{52\mu - \sum^{k}(\mu - 51E)}{52-k} = \mu + \frac{51}{52-k}\sum^{k} E.$$ Hence

the removal and replacement of one "blocked" card by a typical unblocked one has altered the full deck total of the payoffs by

$$\left\{\frac{51}{52-k}+\frac{51}{k}\right\}\sum^{k} E = \frac{52\cdot 51}{k}\,\frac{\sum^{k} E}{(52-k)}.$$

Dividing this result by the previous one, we come up with

$$\frac{52}{(52-k)}\cdot\frac{\sum^{k} E}{k}\cdot\frac{\sum^{13} Y^2}{\sum^{13} YE} \quad \text{as the}$$

adjustment of the primary running count for each extra or deficient blocked card.

Going back to the 14 v T example, the blocked cards are

the k =4 sevens, $\frac{\sum^{k} E}{k} = -4.21,\ \sum^{13} Y^2 = 8$ is the sum

of squares for the primary count, and

$$\sum^{13} YE = .17 - .26 - .77 - 1.41 - 4 \times (1.28) = -7.39.$$

Combining gives $\frac{52\ (-4.21)\cdot 8}{48\ (-7.39)} = 4.9$ points as a more accurate adjustment for each extra or deficient seven in the deck. It is unrealistic to suppose that such auxiliary point values would be remembered more precisely than to the nearest whole number.

D.

Blocks of cards, like $\{6,7,8,9\}$, can be assigned a (4 4 4 4 4 -9 -9 -9 -9 4) count for analytic purposes. Similarly for $\{6,7,8\}$ we would use (3 3 3 3 3 -10 -10 -10 3 3) and for $\{6,7\}$ (2 2 2 2 2 -11 -11 2 2 2). These also will be independent of a primary count which assigns value zero to them, and hence the square root of the sum of the squares of the correlations can be used to find multiple correlation coefficients.

In fact, the original Heath count recommended keeping two counts, what we now call the Gordon (0 1 1 1 1 0 0 0 0 −1) and a "middle against tens" count (0 0 0 0 0 1 1 1 1 −1). These are dependent, having correlation .50, and the more general formula for multiple correlation must be resorted to:

$$\rho = \sqrt{\frac{\rho_{12}^2 + \rho_{13}^2 - 2\rho_{12}\rho_{13}\rho_{23}}{1 - \rho_{23}^2}}$$

I evaluated the system both ways, getting .736 this way and .744 assuming a side count of the block {6,7,8,9} was kept. There is a subtle difference in the information available from the two approaches which justifies the difference.

E.

Factoring in information from cards already included in, and hence dependent upon, the primary count is usually very difficult to do, and probably not worthwhile. One case where it works out nicely, however, is in adjusting the Hi Opt I count by the difference of sixes and fives, for playing 16 v Ten. Both these denominations are included in the primary count, but since it's their difference we are going to be using, our auxiliary count can be taken as (0 0 0 0 1 −1 0 0 0 0) which is uncorrelated with, and effectively independent of, the primary count. The Chapter Eleven simulations contain data on how well this works out.

Even though it is usually too cumbersome in practice to use multiple correlation with dependent counts, an example will establish the striking accuracy of the method. It will also illustrate the precise method of determining the expected deck composition subject to certain card counting information.

Let our problem be the following: there are 28 cards left in the deck and a Ten Counter and Hi Lo player pool information.

The Ten Counter reports there are exactly seven tens remaining, while the Hi Lo player knows there is one more small card (2,3,4,5, and 6) than high card (aces and tens). How many aces should we presume are left in the deck? The Ten Count suggests more than normal, the Hi Lo indicates slightly less than usual.

We can look at this as a multiple regression problem. Let X_1 be the indicator count for aces (−12 1 1 1 1 1 1 1 1 1); X_2 the Ten Count (4 4 4 4 4 4 4 4 4 −9); and X_3 the HiLo (−1 1 1 1 1 1 0 0 0 −1). Hence ρ_{12}, the correlation between X_1 and X_2, is

$$\frac{-52}{\sqrt{156(468)}} = -.19;$$ ρ_{13}, the correlation

between X_1 and X_3, is $$\frac{13}{\sqrt{156(10)}} = .33;$$

while $$\rho_{23} = \frac{52}{\sqrt{10(468)}} = .75.$$

Our correlation matrix is

$$\sum = \begin{pmatrix} 1.00 & -.19 & .33 \\ -.19 & 1.00 & .75 \\ .33 & .75 & 1.00 \end{pmatrix},$$

and the matrix of regression coefficients for predicting the standardized X_1 from the standardized X_2 and X_3 is

$$\sum_{12} \cdot \sum_{22}^{-1} = (-.19, .33)\begin{pmatrix} 1.00 & .75 \\ .75 & 1.00 \end{pmatrix}^{-1} = (-1.00, 1.08).$$

A Ten Count of —21 = —4x21 + 9x7 points with 28 cards in the deck has a standard score of

$$\frac{-21}{\sqrt{\dfrac{28(468)(52-28)}{13(52-1)}}} = -.96, \qquad \text{while}$$

for the Hi Lo we have

$$\frac{-1}{\sqrt{\dfrac{28(10)(52-28)}{13(52-1)}}} = -.31$$

Hence the predicted standard score for the ace is

$$(-1.00, 1.08)\begin{pmatrix}-.96\\-.31\end{pmatrix} = .62$$

This translates into an ace indicator point total of

$$.62\sqrt{\frac{28(156)(52-28)}{13(52-1)}} = 7.8$$

Solving 12a—(28—a) = 7.8, we get a = 2.75 as the predicted number of aces.

The exact distribution can be found by combinatorial analysis for the 21 cards we are uncertain about.

Possible Subset				Number of Possibilities	Probability
Aces	Small (2–6)	Middle (7–9)			
1	9	11	$\binom{4}{1}\binom{20}{9}\binom{12}{11} =$	8,062,080	.0091
2	10	9	$\binom{4}{2}\binom{20}{10}\binom{12}{9} =$	243,877,920	.2759
3	11	7	$\binom{4}{3}\binom{20}{11}\binom{12}{7} =$	532,097,280	.6021
4	12	5	$\binom{4}{4}\binom{20}{12}\binom{12}{5} =$	99,768,240	.1129
			Total	883,805,520	

So the expected number of aces is precisely

$$.0091 + 2(.2759) + 3(.6021) + 4(.1129) = 2.8188$$

The 2.8 aces under the conditions proposed cause me some chagrin, since I had previously constructed an interesting paradox based on what I had thought was an innocuous assumption about their distribution. I had imagined two aces, ten small cards, and nine middle cards would be representative, but we see the precise average figures are 2.8, 10.8, and 7.4, and this destroys my paradox. The only consolation I have is that it was the multivariate methodology which tipped me off to my foolishness.

F.

At no time during the test was any attention paid to whether, in the actual play of the cards, the hand was won or lost. Had the results been scored on that basis, the statistical variation in a sample of this size would have rendered them almost meaningless. The estimate, that perfect play gains 3.66% for decisions made with between 8 and 12 cards remaining, has a standard error of .43%; the other categories have proportionately smaller standard errors.

6

TABLES AND APPLICATIONS

"But what did (the odds) matter to me?. . . .
I wanted to astonish the spectators by taking senseless chances. . ."

Dostoevsky, **The Gambler**

Many players are fascinated by the idea of perfect insurance betting, so the following table should be of interest to them. The player's exact gain at any deck level is catalogued completely for a single deck and extensively for two and four decks. If the remaining number of cards is a multiple of three, add one to it before consulting the charts. For example, with 36 cards left, the single deck gain is the same as with 37, namely .0010%.

The .0010 figure can be interpreted as the gain on the first round for one player looking at seven hands since he is able to see 15 cards. There would be 89 unseen cards at a double deck, and full table, first round insurance is worth only .0001. There aren't any good insurance bets off the top of a four deck shoe, since 64/193 is less than 1/3.

You can also use these tables to get a reasonable estimate for the total profit available from all variations in strategy, not just insurance. Multiply the insurance gain at the number of unplayed cards you're interested in by seven and that should be reasonably close.

Table of Exact Gain From Perfect Insurance (in 1/100 of a %)

One Deck # Cards Left	One Deck Gain	Two Decks # Cards Left	Two Decks Gain	Four Decks # Cards Left	Four Decks Gain
1	241	2	156	4	98
2	157	4	98	8	63
4	98	8	63	16	39
5	85	10	54	20	34
7	68	14	42	28	26
8	62	16	38	32	23
10	52	20	32	40	19
11	48	22	29	44	17
13	42	26	25	52	14
14	40	28	23	56	13
16	35	32	20	64	11
17	33	34	19	68	10
19	30	38	17	76	9
20	28	40	16	80	8
22	25	44	14	88	7
23	24	46	13	92	6
25	21	50	11	100	5.3
26	20	52	10	104	4.9
28	18	56	9	112	4.0
29	17	58	8	116	3.6
31	15	62	7.2	122	3.1
32	14	64	6.6	128	2.6
34	12	68	6	136	2.0
35	11	70	5	140	1.8
37	10	74	4.0	148	1.3
38	9	76	3.5	152	1.1
40	7	80	3	160	.7
41	6	82	2	164	.6
43	5	86	1.5	172	.3
44	4	88	1.1	176	.2
46	2.3	92	.5	184	.05
47	1.7	94	.2	188	.02

Insurance and Betting Effects

Insurance and betting correlation figures can be calculated from the following table of effects of removal of a single card on insurance, Las Vegas basic strategy, and Reno basic strategy expectations. The figures are in %.

EFFECTS OF REMOVAL

	A	2	3	4	5	6	7	8	9	T	Mean	Sum of Squares
Insurance	1.81	1.81	1.81	1.81	1.81	1.81	1.81	1.81	1.81	-4.07	-7.69	95.7
Las Vegas Expectations	-.61	.38	.44	.55	.69	.46	.28	-.00	-.18	-.51	.02	2.84
Reno Expectations	-.54	.37	.44	.55	.65	.46	.25	-.01	-.19	-.50	- .45	2.65

Determination of a card counting system's betting correlation has already been explained in Chapter Four, and insurance correlations are done the same way. Nevertheless, finding the Hi-Lo system's (−1 1 1 1 1 1 0 0 0 −1) insurance correlation will provide a helpful review. Remember, the correlation is the sum of the product of the point values and the effects divided by the square root of the product of the sums of the squares of each set of numbers.[C]

$$\frac{\sum_{i=1}^{13} P_i E_i}{\sqrt{\sum_{i=1}^{13} P_i^2 \cdot \sum_{i=1}^{13} E_i^2}} =$$

$$\frac{5 \times 1 \times 1.81 - 1 \times 1.81 + 4 \times 4.07}{\sqrt{10 \times 95.7}} = \frac{23.52}{30.9} = .76$$

We can use the table in other ways also. Suppose we see a Reno dealer burn a 2 and a 7. What is our approximate expectation? Add .37 to .25 and get .62. Multiply this by 51/50 to get .63. Now, the —.45 in the column entitled "mean" is the full deck house advantage under Reno rules, so we adjust, —.45 + .63 = .18, and estimate a .18% player edge.

The multiplication by 51/50 may surprise you. Here's the story. If we want to know the effect of removing one card from the deck we merely read it directly from the table. However, an extra factor of 51/(52—n) is necessary to adjust the sum of the effects when n cards are removed.

To practice this, let's find the insurance expectation when the dealer's ace and three other non-tens are removed from the deck. We adjust the full deck mean of —7.69 by

$$\frac{51}{48}(1.81 + 1.81 + 1.81 + 1.81) = 7.69$$

and the expectation for the insurance bet is exactly zero, as it should be for a 48 card deck containing 16 tens.

The corresponding effects of removing cards from two or four decks are very nearly one half or one fourth, respectively, of the single deck figures in the table, and if n cards are removed our extra factors become 103/(104—n) and 207/(208—n) respectively. The full deck expectations for basic strategy are different, however, and this is discussed in Chapter 8.[A,B]

Virtually Complete Strategy Tables!

Very lengthy tables are necessary for a detailed analysis of variations in strategy, and a set as complete as any but the antiquarian could desire will follow. In order to condense the printing, the labeling will be abbreviated and uniform throughout the next several pages. Each row will present the ten effects of removal for the cards Ace through Ten, full deck favorability, m, and sum of squares of effects of removal, ss, for the particular strategy variation considered.

For hard totals of 17 down to 12 we are charting the favorability of drawing over standing, that is, how much better off we are to draw to the total than to stand with it. Naturally this will have a negative mean (in the eleventh column) in many cases, since standing is often the better strategy for the full deck.

For hard totals of 11 down to 7 we give the favorability of doubling over merely drawing. Again, in many cases the average favorability for the full deck will be negative, indicating the play is probably not basic strategy. Similarly we present figures for soft doubles, descending from (A,9) to (A,2), showing how much better doubling is than conventional drawing strategy.

Finally, the advantage of pair splitting over not splitting will be catalogued. Not all dealer up cards will have the same set of strategic variations presented, since in many situations (like doubling small totals and soft hands v 9,T, or A and splitting fives) there is no practical interest in the matter.

The tables will be arranged by the different dealer up cards and there will be a separate section for the six and ace when the dealer hits soft 17. (There is no appreciable difference in the Charts for 2,3,4, and 5 up in this case.) Blocks of rows are to be read in descending order: if the heading states "Soft Double (A,9) — (A,2)", there will be eight rows, the top one being for soft doubling 20, the bottom one for 13.

It's important to remember that the entries in the tables are not expectations, but rather differences in expectation for two separate actions being contemplated. Once the cards have been dealt the player's interest in his expectation is secondary to his fundamental concern about how to play the hand. This is resolved by the difference in expectation for the contemplated alternatives.

As a specific example of how to read the table, the arrow on page 76 locates the row corresponding to hard 14 v Ten. The entry in the 11th column, 6.64, is how much (in percent) better it is to draw one and only one card to the total of 14. The entry −4.21 in the seventh column of this row indicates that removal of a single seven makes drawing to 14 less favorable by 4.21%; with one seven removed, the advantage drawing to 14 in the 51 card deck would become 6.64 −4.21 = 2.43%. The entry 27.5 in the 12th column is the sum of the squares of the first nine entries plus four times the square of the tenth entry.

DEALER ACE

HITTING 17-12

−0.53	−1.54	−2.48	−3.09	0.48	0.61	−1.39	−0.37	0.57	1.93	−8.89	35.7
−0.02	−0.92	−1.66	−2.53	−3.18	−1.43	−0.41	0.53	1.40	2.06	13.80	40.2
0.37	−0.15	−0.88	−1.76	−2.61	−4.93	−0.23	0.67	1.48	2.01	16.49	52.6
0.41	0.26	−0.13	− .99	−1.78	−4.17	−3.79	0.79	1.55	1.96	19.09	53.8
0.45	0.28	0.25	− .19	−0.95	−3.19	−3.09	−2.83	1.62	1.91	21.62	46.2
0.46	0.30	0.32	.25	−0.12	−2.22	−2.17	−2.19	−2.05	1.85	24.10	33.2

DOUBLING 11-10

1.74	1.78	1.93	2.21	2.70	2.82	1.60	0.32	-1.02	-3.52	−0.85	83.4
-2.13	1.72	1.84	2.20	2.74	3.22	2.12	0.84	0.09	-3.17	−6.73	78.9

HITTING SOFT 18

-0.41	-1.22	-2.16	0.03	-0.21	1.22	0.03	-0.87	0.22	0.84	0.06	11.5

SPLITTING (99)(88)(77)(66)(33)(AA)

-2.90	-0.94	0.78	0.73	0.66	1.54	0.31	0.59	2.41	-0.80	−4.48	22.0
-2.65	-0.68	-0.05	2.97	3.71	-2.49	0.40	2.29	3.44	-1.73	15.65	65.5
-2.00	-1.64	-2.05	-1.51	1.48	1.80	5.57	0.87	1.39	-0.98	−23.09	56.2
-1.47	-1.45	-2.80	-4.20	-4.41	1.73	2.51	4.23	5.69	0.04	−27.76	108.7
-1.00	0.59	2.06	1.77	0.83	-3.24	-3.55	-2.92	2.44	0.76	−17.93	49.3
2.71	2.88	3.11	3.40	3.97	4.30	3.20	1.79	-0.18	-6.29	16.95	243.0

A playable ace busts only 17% of the time and allows few opportunities for standing with 12-16 as the 11th column shows. For the same reason doubling down is not very advantageous, even with a total of 11.

DEALER ACE
(HIT SOFT 17)

HITTING 17-12

−0.61	−1.67	−2.66	−3.32	0.43	0.39	−1.27	−0.20	0.79	2.02	−7.44	39.3
−0.11	−0.87	−1.74	−2.63	−3.19	−0.05	– .58	0.42	1.33	1.86	4.70	35.7
0.11	−0.24	−0.86	−1.75	−2.54	−3.60	– .38	0.56	1.41	1.83	7.90	37.4
0.17	0.00	−0.24	−0.88	−1.63	−2.87	−3.90	0.68	1.49	1.79	11.05	41.5
0.23	0.04	−0.02	−0.20	– .72	−1.92	−3.12	−2.89	1.56	1.76	14.12	36.9
0.26	0.09	0.07	0.07	– .05	– .97	−2.11	−2.17	−2.07	1.71	17.12	26.5

DOUBLING 11-10

1.56	1.66	1.88	2.30	2.45	2.37	1.90	0.50	−0.98	−3.41	2.60	77.1
−1.96	1.56	1.82	2.33	2.47	2.54	2.19	1.05	0.15	−3.04	−4.39	70.3

HITTING SOFT 18

−0.44	−1.46	−2.37	−0.09	−0.26	0.29	0.15	−0.74	0.40	1.13	5.77	14.0

SPLITTING (99)(88)(77)(33)(AA)

−3.09	−1.19	0.87	0.98	0.61	1.32	0.21	0.51	2.63	−0.71	−2.79	24.1
−2.50	−0.61	0.15	3.33	3.59	−0.73	−0.21	2.18	2.66	−1.97	2.39	58.6
−1.67	−1.32	−1.92	−1.23	1.40	2.59	5.56	0.51	0.95	−1.22	−28.11	56.5
−1.00	0.56	2.36	2.08	1.10	−2.31	−3.66	−3.15	2.11	0.48	−23.18	46.5
2.52	2.78	3.16	3.46	3.67	3.71	3.55	2.06	−0.04	−6.21	20.69	234.3

When the dealer is compelled to hit soft 17, the chance of busting the ace rises to 20%. The 11th column entries for 12-16 are all at least 6.00 less than those on the previous page. Because of the increase in busts and fewer 17's produced, standing and doubling both grow in attractiveness. Note soft 18 is now a profitable hit.

DEALER TEN

HITTING 17-12

-0.86	-0.66	-1.74	-2.53	1.24	1.47	1.04	-0.58	0.06	0.65	−16.93	17.4
-0.49	-0.29	- .80	-1.73	-2.57	1.65	-0.71	-0.06	0.55	1.12	− .45	19.1
-0.17	0.19	-0.32	-0.73	-1.75	-2.23	-0.54	0.09	0.66	1.20	3.11	14.8
-0.08	0.44	0.17	-0.26	-0.77	-1.41	-4.21	0.22	0.77	1.28	6.64	27.5
0.00	0.45	0.40	0.20	-0.26	-0.43	-3.22	-3.48	0.88	1.36	10.13	31.2
0.10	0.46	0.40	0.46	0.24	0.03	-2.06	-2.52	-2.86	1.43	13.58	28.0

DOUBLING 11-10

1.64	0.88	0.89	0.95	1.23	0.69	1.53	0.73	-0.67	-1.97	5.80	26.0
-1.80	0.68	0.67	0.76	1.03	0.72	1.64	0.84	-0.10	-1.11	−2.90	14.6

HITTING SOFT 18

-0.33	-1.01	-1.99	0.48	0.37	0.40	1.43	0.67	-0.31	0.08	3.53	8.2

SPLITTING (88)(33)(22)(AA)

-1.93	-1.79	-2.23	0.13	0.09	0.05	-3.16	-0.55	1.80	1.90	4.41	39.9
-0.78	0.17	1.58	0.85	0.14	-2.45	-4.40	-4.17	1.18	1.97	−22.80	63.6
-1.02	0.56	-0.23	0.31	0.37	0.26	-0.33	-3.48	-3.38	1.73	−23.02	37.3
2.56	2.01	2.25	2.18	2.58	1.96	2.79	2.65	1.01	-5.00	23.97	146.6

It is worth remarking on the magnitude of importance of the 7, 8, and 9 when contemplating standing with 14, 13, and 12 against a Ten. Not only are they desirable cards for the player to draw, but their removal produces the greatest increase in the dealer's chance of busting. The table also shows that soft 18 with no card higher than a 3 should not be hit.

DEALER 9

HITTING 17-12

−1.47	−1.81	−2.13	−2.51	1.31	1.56	1.79	1.40	−0.10	0.49	−13.29	26.6
−0.79	−0.90	−1.25	−2.10	−2.48	1.60	1.96	−0.22	0.39	0.95	2.97	23.7
−0.40	−0.34	−0.29	−1.14	−2.09	−2.22	1.92	−0.07	0.50	1.03	6.53	19.1
−0.25	−0.01	0.21	−0.17	−1.14	−1.88	−1.91	0.07	0.61	1.12	10.07	13.5
−0.10	0.10	0.47	0.30	−0.19	−0.93	−1.57	−3.60	0.72	1.20	13.57	22.6
0.05	0.20	0.48	0.55	0.32	0.03	−0.65	−3.10	−2.98	1.27	17.04	26.1

DOUBLING 11-10

2.60	2.07	1.18	1.34	1.65	0.88	-0.16	0.10	-1.29	-2.10	7.54	37.0
-0.41	2.17	1.15	1.36	1.68	1.14	0.24	0.89	-0.51	-1.93	2.72	28.2

HITTING SOFT 18

-1.16	-1.49	-1.86	0.34	0.20	0.56	0.79	1.82	1.05	-0.06	8.54	12.5

SPLITTING (9 9)(8 8)(3 3)(2 2)(A A)

-2.17	-0.19	1.16	0.92	0.75	1.26	1.63	2.10	-0.04	-1.36	8.81	23.5
-1.10	-1.18	-2.03	0.17	-0.02	-0.08	0.63	-2.56	0.01	1.54	14.30	23.2
0.15	1.04	1.80	1.50	0.63	-2.97	-4.34	-4.60	0.71	1.52	−16.05	65.5
-0.04	1.17	-0.13	0.31	0.58	0.69	-0.42	-4.26	-3.62	1.43	−16.18	41.9
3.32	3.19	2.48	2.50	2.93	2.12	1.49	1.83	-0.19	-4.92	22.47	148.9

A 9 behaves very much like a Ten, except that the dealer's totals gravitate toward 19 rather than 20. Since 19 is easier to beat, the player is inclined to hit and double down more often than against a Ten.

DEALER 8

HITTING 17-12

−2.29	−2.34	−2.86	−2.48	1.38	1.66	1.84	2.18	1.78	0.28	−12.34	41.0
−1.05	−1.39	−2.44	−2.20	−2.36	1.72	1.93	2.38	0.20	0.80	5.23	34.3
−0.63	−0.31	−1.35	−1.76	−2.05	−2.04	1.91	2.32	0.33	0.90	9.03	26.2
−0.46	0.04	−0.26	−0.74	−1.61	−1.77	−1.90	2.29	0.45	0.99	12.81	19.3
−0.29	0.15	0.12	0.28	−0.61	−1.39	−1.62	−1.53	0.57	1.08	16.56	12.0
−0.13	0.26	0.25	0.56	0.39	−0.39	−1.23	−1.30	−3.09	1.17	20.27	18.8

DOUBLING 11-9

2.75	2.11	2.53	1.71	1.98	1.10	-0.38	-2.09	-1.36	-2.09	12.62	50.3
-0.64	2.25	2.66	1.77	2.05	1.46	0.55	-1.18	-1.10	-1.95	8.82	40.2
-0.37	2.89	2.32	1.43	1.65	0.90	0.38	-0.81	-0.43	-1.99	−12.78	36.2

HITTING SOFT 18

-2.20	-1.68	-2.21	0.09	-0.04	0.37	0.70	1.14	2.02	0.45	−6.73	19.4

SOFT DOUBLING 18

-1.97	-2.77	-3.07	1.76	1.85	1.53	1.29	0.87	2.19	-0.42	−13.99	37.8

SPLITTING (99) (77)(66)(33)(22)(AA)

-3.61	-0.14	1.84	1.14	1.06	1.61	2.05	2.28	0.54	-1.69	9.03	42.6
-0.44	-0.99	-1.12	-1.57	1.50	2.05	2.61	-1.88	-1.49	0.33	− 5.13	24.3
-0.91	-1.43	-2.96	-4.55	-4.52	2.13	2.91	3.52	3.85	0.49	−18.45	94.0
-0.11	1.47	2.55	1.72	0.18	-3.39	-4.12	-3.97	0.41	1.32	− 4.58	63.0
-0.31	0.72	0.81	0.87	1.19	0.28	-0.85	-3.69	-3.91	1.22	− 5.45	39.1
3.51	3.00	3.56	2.87	3.22	2.75	1.39	-0.96	-0.35	-4.75	25.06	153.2

The fact that hard 17 is the most volatile of the stiff hitting situations is revealed by the 12th column figure of 41.0. A player who split three eights and drew (8,9), (8,7.9), and (8,9) would be more than 5% better off to hit the last total of 17 even though the hand was dealt from a full pack!

DEALER 7

HITTING 17-12

−3.65	−2.84	−3.31	−3.72	0.78	1.17	1.47	1.67	2.01	1.61	−37.79	67.3
−1.88	−1.93	−2.44	−2.78	−2.33	1.80	2.10	2.32	2.77	0.59	6.07	48.2
−0.95	−0.82	−1.89	−2.33	−1.96	−1.94	2.12	2.29	2.71	0.70	10.17	37.2
−0.76	0.03	−0.75	−1.76	−1.59	−1.61	−1.68	2.25	2.65	0.81	14.25	27.1
−0.58	0.15	0.12	−0.61	−1.10	−1.29	−1.40	−1.53	2.60	0.91	18.30	17.8
−0.39	0.26	0.25	0.28	−0.04	−0.85	−1.08	−1.25	−1.23	1.01	22.31	9.1

DOUBLING 11-8

2.88	2.26	2.59	3.03	2.29	0.84	−0.85	−1.84	−2.90	−2.08	18.18	64.9
−0.45	2.27	2.71	3.19	2.37	1.69	0.10	−1.48	−2.67	−1.93	14.01	55.7
−0.51	2.97	2.58	2.98	2.03	1.22	0.50	−1.02	−2.88	−1.97	− 6.51	55.3
0.05	2.65	3.24	2.70	1.66	0.74	−0.02	−0.73	−2.55	−1.93	−27.29	50.0

HITTING SOFT 17

−1.97	−1.67	−2.16	−2.62	−0.30	0.26	0.82	1.28	1.85	1.13	15.70	29.1

SOFT DOUBLING (A7)(A6)

−1.08	−2.47	−2.81	2.97	2.20	1.87	1.52	1.30	−0.60	−0.72	−19.06	38.8
0.65	−0.58	−0.38	−0.16	2.29	1.42	0.57	−0.08	−0.98	−0.69	− 6.96	11.4

SPLITTING (99)(66)(33)(22)(AA)

−3.48	−1.29	1.13	1.27	0.70	1.35	1.82	1.14	−0.31	−0.58	−6.40	25.1
−1.82	−2.23	−3.03	−3.69	−3.73	2.57	3.09	3.69	4.36	0.20	−8.76	94.0
−0.20	1.69	2.12	1.88	−0.22	−3.13	−3.76	−3.65	1.07	1.05	5.08	53.8
−0.69	0.16	1.14	2.14	0.79	−0.12	−0.63	−3.28	−3.26	0.94	4.39	32.3
3.84	3.20	3.55	4.15	3.77	2.56	0.29	−0.88	−2.05	−4.61	29.94	165.4

Standing with 12 against a 7 will almost never be justified as we can see from the large value of m in the 11th column and small value of ss in the 12th column. Note that otherwise the 9 is almost always a more important high card than the Ten.

DEALER 6

HITTING 17-12

−2.12	−3.56	−4.08	−4.50	−0.75	1.07	1.39	1.61	1.92	2.25	−52.09	84.5
−3.72	−2.96	−3.40	−3.77	−4.08	1.35	1.69	1.92	2.24	2.68	−27.29	107.5
−3.12	−2.19	−2.73	−3.08	−3.40	−2.24	1.80	2.01	2.27	2.67	−20.27	88.9
−2.78	−1.62	−1.92	−2.37	−2.68	−1.74	−1.85	2.04	2.27	2.66	−13.73	71.3
−2.43	−1.35	−1.31	−1.53	−1.94	−1.21	−1.39	−1.64	2.31	2.66	− 7.25	55.5
−2.10	−1.10	−1.04	−0.92	−1.11	−0.71	−0.94	−1.20	−1.40	2.63	− .76	41.4

DOUBLING 11-7

1.71	1.61	1.90	2.25	2.88	0.68	-0.14	-0.77	-1.42	-2.17	33.86	44.5
-0.49	1.44	1.74	2.37	3.04	0.93	0.29	-0.46	-1.15	-1.93	28.91	37.6
-0.23	2.15	1.79	2.46	2.80	0.53	0.19	-0.37	-1.20	-2.03	12.00	40.2
0.25	2.18	2.93	2.28	2.47	0.13	-0.24	-0.49	-1.11	-2.10	− 3.10	43.9
1.35	2.26	2.69	2.94	2.13	-0.28	-0.68	-0.94	-1.23	-2.06	−17.25	47.4

SOFT DOUBLING (A9)–(A2)

-0.49	3.28	3.90	3.92	3.89	1.65	0.55	-0.88	-2.11	-3.42	−12.34	111.7
0.31	-0.87	3.19	3.12	3.62	1.44	1.05	0.07	-1.34	-2.65	− 1.77	66.9
1.24	-0.82	-1.71	2.83	3.37	1.25	0.90	0.63	-0.32	-1.84	9.01	41.0
0.63	0.34	0.31	0.31	2.75	0.56	0.19	-0.08	-0.40	-1.15	11.81	14.1
1.78	0.77	0.73	0.73	0.57	0.40	0.02	-0.27	-0.59	-1.03	7.29	10.0
2.37	1.54	1.21	1.22	1.04	-1.75	-0.06	-0.35	-0.68	-1.14	6.15	20.9
2.47	2.16	1.98	1.70	1.52	-1.29	-2.24	-0.46	-0.79	-1.26	4.26	33.8
2.58	2.22	2.60	2.46	1.97	-0.84	-1.80	-2.66	-0.91	-1.41	2.24	48.1

SPLITTING (99)(44)(33)(22)

1.18	0.43	2.54	3.74	4.32	1.65	1.12	-0.04	-2.24	-3.18	13.03	90.1
3.22	2.29	2.12	1.48	1.21	-1.88	-2.30	-0.62	-0.96	-1.14	−10.96	39.1
1.12	1.80	2.09	2.97	3.00	-1.09	-1.97	-2.30	-0.65	-1.24	7.62	43.6
0.83	0.88	1.94	2.69	3.12	0.28	-0.65	-2.60	-3.03	-0.87	8.18	41.6

In multiple deck play doubling with (A,8) v 6 is just fractionally the wrong thing to do. On the next page it will be seen to be the correct play when dealer hits soft 17, paradoxically even though the player's hand has a smaller expectation whether doubled or undoubled.

DEALER 6
(HIT SOFT 17)

HITTING 17-12

−2.32	−3.61	−4.16	−4.60	−0.77	1.08	1.43	1.68	2.02	2.31	−51.38	88.9
−2.72	−3.04	−3.50	−3.89	−4.15	1.30	1.67	1.93	2.28	2.53	−31.79	100.1
−2.23	−2.34	−2.77	−3.15	−3.43	−2.27	1.78	2.01	2.30	2.52	−24.60	83.1
−1.94	−1.84	−2.04	−2.38	−2.66	−1.73	−1.85	2.05	2.30	2.53	−17.81	65.9
−1.65	−1.57	−1.51	−1.62	−1.90	−1.17	−1.37	−1.63	2.32	2.52	−11.05	50.7
−1.37	−1.29	−1.22	−1.08	−1.13	−0.62	−0.86	−1.14	−1.36	2.52	− 4.33	37.3

DOUBLING 11-7

1.68	1.57	1.90	2.32	2.80	0.67	-0.09	-0.77	-1.48	-2.15	33.73	43.8
-0.37	1.38	1.74	2.45	2.95	0.81	0.25	-0.44	-1.18	-1.90	28.41	36.5
-0.12	2.07	1.85	2.55	2.71	0.41	0.04	-0.45	-1.22	-1.96	11.70	38.8
0.35	2.16	2.93	2.37	2.41	0.02	-0.40	-0.68	-1.24	-1.98	− 3.00	42.6
1.09	2.31	2.75	3.02	2.13	-0.33	-0.78	-1.09	-1.42	-1.92	−15.09	46.4

SOFT DOUBLING (A9)–(A2)

-0.77	3.31	3.91	3.92	3.77	1.50	0.55	-0.74	-2.08	-3.34	−10.83	108.5
-0.12	-0.73	3.14	3.13	3.50	1.28	0.85	0.05	-1.23	-2.47	0.77	60.7
0.58	-0.68	-1.58	2.85	3.25	1.09	0.71	0.42	-0.36	-1.57	12.26	33.8
0.82	0.49	0.45	0.47	2.67	0.44	0.04	-0.26	-0.60	-1.13	11.65	14.2
1.68	0.97	0.94	0.94	0.61	0.32	-0.09	-0.41	-0.76	-1.05	8.20	11.2
2.10	1.68	1.45	1.46	1.11	-1.74	-0.17	-0.49	-0.84	-1.14	7.09	21.9
2.16	2.13	2.16	1.97	1.62	-1.25	-2.26	-0.58	-0.95	-1.25	5.34	34.5
2.22	2.19	2.60	2.67	2.11	-0.76	-1.78	-2.70	-1.06	-1.37	3.49	47.8

SPLITTING (99)(44)(33)(22)

0.12	0.53	2.63	3.88	4.27	1.57	1.03	0.08	-2.20	-2.98	17.96	84.4
2.22	2.54	2.45	1.77	1.53	-1.62	-2.11	-0.59	-1.05	-1.28	− 6.08	38.0
0.98	1.90	2.17	3.25	3.15	-0.92	-1.85	-2.23	-0.76	-1.42	8.75	47.7
0.88	1.15	2.24	3.01	3.35	0.54	-0.42	-2.34	-2.83	-1.39	8.67	49.1

When the dealer hits soft 17 the 6 breaks almost 2% more often than otherwise. Standing and soft doubling become more frequent activities. As mentioned on the previous page, (A, 8) is a basic strategy double down, regardless of the number of decks used.

DEALER 5

HITTING 17-12

-1.86	-2.53	-4.15	-4.66	-0.92	-0.43	1.39	1.75	2.01	2.35	-47.85	80.8
-1.37	-2.99	-3.51	-3.95	-4.29	-0.01	1.59	1.96	2.24	2.58	-28.54	95.0
-0.97	-2.29	-2.79	-3.22	-3.56	-3.44	1.70	2.03	2.29	2.56	-21.64	87.2
-0.76	-1.82	-2.06	-2.47	-2.80	-2.75	-1.91	2.10	2.30	2.54	-14.79	69.0
-0.58	-1.57	-1.55	-1.70	-2.01	-2.02	-1.39	-1.59	2.29	2.53	- 8.38	52.0
-0.40	-1.32	-1.28	-1.17	-1.22	-1.27	-0.88	-1.13	-1.39	2.52	- 1.98	37.6

DOUBLING 11-7

1.33	1.34	1.64	2.01	2.51	2.35	-0.02	-0.82	-1.50	-2.21	31.44	44.5
-0.81	1.15	1.43	1.89	2.68	2.73	0.36	-0.47	-1.19	-1.94	26.13	39.1
-0.70	1.81	1.27	2.02	2.73	2.43	0.16	-0.46	-1.22	-2.01	8.80	40.7
-0.39	1.61	2.42	2.17	2.48	2.04	-0.30	-0.69	-1.22	-2.03	- 6.63	42.2
-0.00	1.77	2.55	2.96	2.17	1.68	-0.71	-1.12	-1.41	-1.97	-19.40	45.3

SOFT DOUBLING (A9)–(A2)

-1.83	2.83	3.52	4.08	4.03	3.42	0.68	-0.77	-2.12	-3.46	-14.78	121.5
-1.22	-1.28	3.30	3.22	3.28	3.17	0.99	0.07	-1.24	-2.57	- 2.85	74.1
-0.57	-0.66	-1.67	2.43	3.04	2.95	0.85	0.46	-0.32	-1.63	9.44	39.0
0.27	-0.09	0.06	0.07	2.57	2.36	0.15	-0.25	-0.56	-1.14	8.62	17.9
0.99	0.60	0.58	0.59	0.43	2.23	-0.00	-0.42	-0.73	-1.07	4.65	12.4
1.42	1.36	1.12	1.13	0.95	0.10	-0.08	-0.50	-0.83	-1.17	3.20	13.7
1.44	1.81	1.87	1.66	1.48	0.61	-2.26	-0.60	-0.93	-1.27	1.81	26.9
1.46	1.88	2.35	2.41	2.00	1.11	-1.76	-2.81	-1.05	-1.40	- 0.04	42.1

SPLITTING (99) (44) (33) (22)

-1.42	1.19	2.61	2.82	4.14	3.87	1.21	0.08	-2.05	-3.11	14.67	94.7
0.63	2.55	2.58	1.62	1.27	0.53	-2.12	-0.63	-1.07	-1.34	-11.09	31.3
0.05	1.52	1.99	2.88	3.02	1.28	-1.84	-2.33	-0.74	-1.46	4.67	43.1
-0.06	0.96	1.83	2.64	3.24	2.80	-0.35	-2.46	-2.90	-1.43	4.61	52.3

The 11th column full deck advantage figures on pages 74-85 come from exact 52 card calculations, without the dealer's up card or any of the player's cards removed. The effects of removal (first 10 columns) are, for hitting totals of

DEALER 4

HITTING 17-12

−1.89	−1.78	−3.09	−4.70	−1.01	−0.61	−0.03	1.71	1.98	2.35	−44.62	68.4
−1.41	−1.52	−3.47	−3.98	−4.35	−0.21	0.35	1.93	2.23	2.61	−24.46	86.8
−1.01	−0.92	−2.74	−3.24	−3.61	−3.64	0.59	1.98	2.25	2.59	−18.32	81.9
−0.84	−0.52	−2.01	−2.50	−2.87	−2.93	−2.89	2.04	2.29	2.56	−11.81	71.9
−0.63	−0.36	−1.53	−1.74	−2.10	−2.21	−2.23	−1.60	2.29	2.53	– 5.38	54.0
−0.44	−0.23	−1.28	−1.21	−1.31	−1.45	−1.53	−1.13	−1.42	2.50	0.65	38.4

DOUBLING 11-7

1.47	0.96	1.46	1.82	2.26	2.10	1.66	-0.73	-1.46	-2.39	28.80	46.2
-0.74	0.80	1.25	1.65	2.18	2.50	2.29	-0.36	-1.13	-2.11	23.25	40.9
-0.62	1.36	1.01	1.52	2.26	2.50	2.21	-0.34	-1.15	-2.19	5.30	42.4
-0.29	0.93	1.93	1.70	2.35	2.18	1.76	-0.56	-1.13	-2.22	−10.80	42.2
0.12	0.65	2.10	2.85	2.14	1.80	1.33	-1.00	-1.32	-2.17	−24.13	44.1

SOFT DOUBLING (A9)–(A2)

-1.83	1.81	3.03	3.69	4.18	3.68	2.53	-0.73	-2.14	-3.55	−19.21	122.3
-1.21	-2.60	2.92	3.38	3.36	2.96	2.84	0.16	-1.24	-2.64	– 6.80	85.8
-0.53	-1.93	-1.67	2.52	2.61	2.75	2.68	0.57	-0.28	-1.68	5.91	46.4
0.38	-0.74	-0.48	-0.26	2.32	2.32	2.09	-0.12	-0.45	-1.27	5.43	22.8
1.14	-0.20	0.24	0.29	0.11	2.20	1.95	-0.29	-0.63	-1.20	1.05	16.3
1.59	0.56	0.81	0.87	0.68	0.01	1.87	-0.38	-0.74	-1.32	– 0.89	15.9
1.62	1.02	1.61	1.43	1.23	0.54	-0.36	-0.48	-0.84	-1.45	– 2.37	19.6
1.65	1.06	2.11	2.22	1.78	1.07	0.15	-2.76	-0.95	-1.58	– 4.01	36.0

SPLITTING (99)(66)(33)(22)

-1.54	-0.44	3.18	2.66	2.92	3.61	3.34	0.00	-2.04	-2.92	11.70	90.7
0.34	0.24	0.78	0.97	1.71	3.51	3.60	0.73	-0.01	-2.97	5.41	65.8
-0.04	0.37	1.68	2.37	2.50	0.97	0.19	-2.48	-0.84	-1.18	0.25	28.3
-0.51	-0.33	0.94	1.76	2.37	2.17	1.37	-2.97	-3.45	-0.34	3.01	37.7

17-12, exact figures obtained by comparing the 11th column figure with the appropriate 51 card deck advantage. However, for doubling and splitting removal effects the amount of computer time necessary to carry out the calculations exactly would have been excessive; in these situations the removal ef-

DEALER 3

HITTING 17-12

−1.91	−1.86	−2.37	−3.64	−1.07	−0.67	−0.23	0.31	2.05	2.35	−41.44	53.4
−1.45	−1.30	−2.07	−3.99	−4.46	−0.30	0.23	0.68	2.27	2.60	−21.34	75.7
−1.01	−0.70	−1.46	−3.25	−3.72	−3.73	0.46	0.88	2.27	2.57	−15.55	73.7
−0.86	−0.34	−0.82	−2.50	−2.95	−3.01	−3.01	1.07	2.27	2.54	− 9.73	66.1
−0.68	−0.19	−0.41	−1.75	−2.19	−2.30	−2.34	−2.45	2.29	2.50	− 3.54	55.7
−0.47	−0.06	−0.25	−1.25	−1.41	−1.56	−1.65	−1.81	−1.41	2.46	2.57	39.0

DOUBLING 11-7

1.56	0.96	0.99	1.61	2.09	1.84	1.40	0.97	−1.45	−2.49	26.44	44.6
−0.75	0.84	0.81	1.44	1.96	2.01	2.05	1.58	−1.12	−2.20	20.72	39.3
−0.62	1.37	0.44	1.24	1.78	2.04	2.28	1.73	−1.13	−2.28	2.10	41.6
−0.28	0.93	1.15	1.17	1.89	2.05	1.90	1.51	−1.10	−2.31	−14.42	39.9
0.15	0.57	0.89	2.37	2.03	1.78	1.45	1.07	−1.28	−2.26	−28.32	39.4

SOFT DOUBLING (A9)–(A2)

−1.83	1.84	2.02	3.20	3.80	3.81	2.73	1.06	−2.19	−3.61	−23.10	115.4
−1.20	−2.76	1.83	2.99	3.54	3.03	2.58	1.97	−1.26	−2.68	−10.15	83.7
−0.50	−2.12	−2.94	2.68	2.72	2.31	2.42	2.40	−0.28	−1.68	3.33	56.3
0.41	−0.78	−1.20	−0.82	2.10	2.08	2.06	1.85	−0.40	−1.32	2.51	26.5
1.20	−0.28	−0.65	−0.08	−0.18	1.97	1.91	1.69	−0.58	−1.25	− 1.93	19.0
1.67	0.50	−0.10	0.54	0.43	−0.28	1.85	1.61	−0.68	−1.38	− 3.94	17.7
1.69	0.97	0.69	1.14	1.03	0.30	−0.44	1.51	−0.79	−1.53	− 6.00	19.2
1.72	0.99	1.17	1.97	1.61	0.86	0.10	−0.81	−0.90	−1.68	− 7.83	25.2

SPLITTING (99)(77)(66)(33)(22)

−1.49	−0.54	1.87	3.27	2.80	2.35	2.92	2.06	−2.08	−2.79	7.51	78.3
−0.74	−0.63	−1.06	−1.08	2.65	2.73	1.93	1.97	−0.84	−1.23	12.61	32.1
0.11	−0.35	−0.53	0.98	1.90	4.10	4.37	4.01	0.65	−3.81	1.06	115.6
0.04	0.45	0.98	1.90	2.02	0.39	−0.21	−0.54	−1.03	−1.00	− 4.00	14.4
−0.24	0.24	0.31	1.00	1.53	1.17	0.48	−1.64	−4.07	0.31	0.50	24.8

fects were estimated by judicious alteration of infinite deck probabilities.

Use of these tables to carry out variations in strategy for the 5,000 hand experiment reported on page 61 resulted in an overall playing efficiency of 98.7%, ranging from 100% with

DEALER 2

HITTING 17-12

−1.92	−1.86	−2.38	−2.93	−0.03	−0.72	−0.30	0.21	0.66	2.32	−38.23	43.4
−1.43	−1.37	−1.83	−2.56	−4.47	−0.37	0.09	0.61	1.06	2.57	−17.67	60.8
−0.98	−0.73	−1.22	−1.96	−3.74	−3.82	0.30	0.79	1.21	2.54	−12.19	62.0
−0.80	−0.36	−0.60	−1.31	−2.98	−3.10	−3.17	0.90	1.35	2.50	− 6.77	58.3
−0.64	−0.26	−0.21	−0.65	−2.22	−2.37	−2.48	−2.54	1.50	2.46	− 1.33	50.1
−0.47	−0.12	−0.07	−0.22	−1.43	−1.63	−1.80	−1.90	−2.05	2.42	4.49	39.9

DOUBLING 11-8

1.71	1.02	1.04	1.11	1.88	1.62	1.09	0.63	0.21	-2.57	23.72	40.5
-0.72	0.94	0.87	0.97	1.75	1.74	1.51	1.26	0.77	-2.27	17.84	34.2
-0.57	1.45	0.52	0.60	1.50	1.50	1.77	1.72	0.87	-2.34	− 1.38	36.4
-0.21	1.01	1.21	0.37	1.35	1.54	1.73	1.57	0.92	-2.37	−18.61	35.7

SOFT DOUBLING (A9)–(A2)

-1.84	1.93	2.04	2.18	3.32	3.40	2.84	1.18	-0.46	-3.65	−27.40	101.6
-1.20	-2.81	1.80	1.89	3.15	3.17	2.65	1.64	0.50	-2.70	−13.94	75.1
-0.48	-2.15	-3.17	1.59	2.88	2.39	1.99	2.11	1.51	-1.67	− 0.10	53.3
0.49	-0.77	-1.22	-1.59	1.65	1.80	1.77	1.73	1.53	-1.35	− 0.86	26.5
1.31	-0.24	-0.69	-1.01	-0.57	1.74	1.64	1.57	1.35	-1.28	− 5.45	20.1
1.80	0.57	-0.13	-0.43	0.07	-0.61	1.63	1.50	1.26	-1.42	− 7.44	18.6
1.82	1.06	0.68	0.15	0.70	0.01	-0.75	1.45	1.16	-1.57	− 9.80	19.3
1.91	1.14	1.24	1.02	1.38	0.68	-0.10	-0.90	1.15	-1.88	−12.42	26.2

SPLITTING (99)(77)(66)(33)(22)

-1.56	-0.58	1.61	2.06	3.47	2.27	1.79	1.73	-0.04	-2.69	4.51	61.9
-0.79	-0.76	-1.00	-1.24	1.36	2.17	1.59	1.76	1.14	-1.06	9.49	21.7
0.08	-0.43	-0.79	-0.95	1.40	3.87	4.02	3.86	3.41	-3.62	− 3.52	113.7
-0.14	0.26	0.92	0.88	1.44	-0.22	-0.88	-1.01	0.75	-0.50	− 7.12	7.2
-0.21	0.34	0.38	0.78	1.16	0.74	-0.00	-1.93	-2.47	0.31	− 2.66	13.0

43-47 cards left to 97.5% with 8-12 cards remaining. The relatively few and inconsequential errors appear more attributable to blackjack's essential non-linearity, which is more pronounced deeper in the deck, than to any approximations in the table.

How to use these Tables

Now, the first thing we can do with these tables is find the correlation of our card counting system for a particular change in strategy. This is done exactly as it was for the insurance and betting effects previously.

Another use is to find some of the "composition" dependent departures from the simplified basic strategy defined in Chapter Two. Should you hit or stand with (4,4,4,4) v 8? To the full deck favorability of 5.23% for hitting 16 v 8 we add 51/47 of the sum of the effects of the four removed 4's and the dealer's 8.

$$\frac{51}{47}(2.38 - 4(2.20)) = -6.97$$

Since $5.23 - 6.97 = -1.74$ is negative we can presume that standing is better by about 1.74%.

In Chapter Two the question was asked whether one should hit (8,2,2,2) v T after having busted (8,7,7) on the first half of a pair split. The table for hard 14 against a ten gives the following estimate for the advantage for hitting in this case

$$6.64 + \frac{51}{44}[3(.44) + 2(-4.21) + 2(.22) + 1.28] = + .40(\%)$$

Incidentally, if doubling down had been allowed after splitting, the quandary would never have arisen for the optimal strategist; he would have doubled with (8,2) since the doubling advantage after (8,7,7) was removed would be about

$$-2.90 + \frac{51}{46}(.68 + 2(1.64 + .84) - 1.11) = 2.12(\%)$$

Just for drill the reader might confirm the 2.3% advantage hitting (6,4,6) v T mentioned in Chapter One and also show that the player is .6% better to stand with (6,4,5) v T, which is interesting because if you draw a 5 to (6,4) and get 15 you should stand, while if you get a 6 you're one step closer to busting but should hit. Don't forget to remove the dealer's up card as well as the cards in the player's hand, since all of these tables assume a 52 card deck from which dealer's and player's cards have not yet been removed. Also don't be surprised if you are unable to reproduce exactly the 2.3% in the first case; after all, these methods are only approximate.

Quantifying the Spectrum of Opportunity at various Points in the Deck

Before we will be able to quantify betting and strategy variations at different points in the deck we'll have to introduce another table, which has an existence of its own, independently of any blackjack considerations, and is called by some the "Unit Normal Linear Loss Integral." No card counting enthusiast should be without it!

First the table itself. Corresponding to values of a variable designated by z, which ranges from 0 to 2.58 in increments of .02, we have arranged the associated values of this special and important mathematical function.

Unit Normal Linear Loss Integral

z	.00	.02	.04	.06	.08
0.00	.3989	.3890	.3793	.3697	.3602
0.10	.3509	.3418	.3329	.3240	.3154
0.20	.3069	.2986	.2904	.2824	.2745
0.30	.2668	.2592	.2518	.2445	.2374
0.40	.2304	.2236	.2170	.2104	.2040
0.50	.1978	.1917	.1857	.1799	.1742
0.60	.1687	.1633	.1580	.1528	.1478
0.70	.1429	.1381	.1335	.1289	.1245
0.80	.1202	.1160	.1120	.1080	.1042
0.90	.1004	.0968	.0933	.0899	.0866
1.00	.0833	.0802	.0772	.0742	.0714
1.10	.0686	.0660	.0634	.0609	.0585
1.20	.0561	.0539	.0517	.0496	.0475
1.30	.0456	.0437	.0418	.0401	.0383
1.40	.0367	.0351	.0336	.0321	.0307
1.50	.0293	.0280	.0268	.0256	.0244
1.60	.0233	.0222	.0212	.0202	.0192
1.70	.0183	.0174	.0166	.0158	.0150
1.80	.0143	.0136	.0129	.0122	.0116
1.90	.0110	.0104	.0099	.0094	.0089
2.00	.0084	.0080	.0075	.0071	.0067
2.10	.0063	.0060	.0056	.0053	.0050
2.20	.0047	.0044	.0042	.0039	.0037
2.30	.0036	.0034	.0032	.0030	.0028
2.40	.0027	.0026	.0024	.0023	.0022
2.50	.0021	.0018	.0017	.0016	.0016

To illustrate the use of the table we will approximate the player's gain from perfect insurance when there are 40 cards left, for one, two, and four decks. The following step by step procedure will be used in all such calculations.

Step

1. Calculate $b = 51\sqrt{\dfrac{ss \cdot (N-n)}{13 \cdot (N-1) \cdot n}}$, where

   ss = the sum of squares of effects of removal,
   n = number of cards remaining in the deck, and
   N = Number of cards originally in the full pack.

   For the single deck case ss = 95.7, N = 52, and n = 40; thus b = 10.6
2. Calculate z = m/b, where m is the full deck average favorability for carrying out the play. Ignore the algebraic sign of m. In our example z = 7.69/10.6 = .72
3. Look up in the UNLLI chart the number corresponding to z. In our case this will be .1381
4. Multiply the number found in step 3 by b. For us 10.6 × .1381 = 1.5. This is the conditional player gain (in %), assuming the dealer does have an ace showing.
5. If desired, adjust the figure found in step 4 to reflect the likelihood that the situation will arise. In our insurance example we multiply by 1/13, the chance the dealer shows an ace, but we also divide by 2, since the insurance bet can only be for one half of the player's original wager. 1.5/26 = .06(%), which is just less than the .07 figure given in the first table of this chapter.

Repeating the procedure, for two decks, we have

1. N = 104, n= 40, $b = 51\sqrt{\dfrac{95.7\,(64)}{13(103)(40)}} = 17.2$
2. z = 7.69/17.2 = .45
3. We would interpolate between .2170 and .2104, let's say .214.
4. (17.2) (.214) = 3.68
5. 3.68/26 = .14 (in %), again slightly on the small side, the correct value being .16.

For four decks,

1. $N = 208$, $b = 19.7$
2. $z = .39$
3. Take .234, between .2374 and .2304
4. 4.61
5. $4.61/26 = .18$, close to .19 from the exact table.

If you're disappointed in the accuracy, there are ways of improving the approximation, principally by adjusting for the dealer's up card. Removing the dealer's ace changes m, for the single deck, to $-7.69 + 1.81 = -5.88$, and N becomes 51, rather than 52. Repeating the calculations, 1. $b = 10.26$; 2. $z = 5.88/10.26 = .57$; 3. .177 from the table; 4. $(.177)(10.26) = 1.816$; 5. $1.816/26 = .07$, in agreement with the precise figure.

How much would perfect knowledge of when to hit hard 12 against a 6 be worth with $n = 5$ cards left in the deck? Using

$N = 51$, we calculate $b = 51\sqrt{\dfrac{41.4\,(46)}{13(50)\cdot 5}} = 39.0$. After

revising m from $-.76$ to -1.49 in order to account for the removal of the dealer's six, we get $z = 1.49/39.0 = .04$, extract the .3793 value from the UNLLI table and multiply this by b to produce our estimate $39.0\,(.3793) = 14.8$ (%). The exact gain in this situation appears in Chapter Eleven and is 15.6 (%).

One thing remains, and that is instruction on how to calculate a card counting system's gain, rather than the gain from perfect play. To do this we must have a preliminary calculation of the correlation of the card counting system and the particular play examined. Since we already found the correlation of the Hi Lo system for insurance to be .76 we will use it as an example.[C]

The only modification in the original five step procedure is in step 1. After calculating b in the usual fashion we then multiply it by the card counting system's correlation coefficient and use the resultant product as a revised value of b in all subsequent calculations. Using the 51 card deck calculation, which is more accurate, we get b = (10.26) (.76) = 7.8 in step 1. Then 2. z = 5.88/7.8 = .75; 3. .131 from the table. 4. (.131) (7.8) = 1.02; 5. 1.02/26 = .04 (in %). Thus the efficiency of the Hi Lo system, at the 40 card level, is .04/.07 = 57% — it would exploit 57% of the gain available from perfect insurance betting.

The Normal Distribution of Probability

The famous normal distribution itself can be used to answer many probabilistic questions with a high degree of accuracy. The table on page 91 exhibits the probability that a "standard normal variable" will have a value between 0 and selected values of z (used to designate such a variable) from 0 to 3.08.

Chance of Being behind

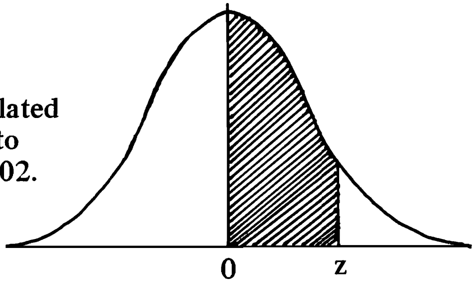
One type of question that can be answered with this table is "Suppose I have an average advantage of 2% on my big bets; What is the chance that I will be behind (on big bets) after making 2500 of them?" Before answering this we will have to borrow ahead from Chapter Eleven, where we learn that the variance of a blackjack hand is about 1.26 squared units. Hence the average, or expected, result for these 2500 hands will be 2% of 2500, or 50 units ahead, and the variance is 2500(1.26) = 3150 squared units. Since we want to find the probability that our profit will be less than zero, we "standardize" zero itself:

$$z = \frac{0-50}{\sqrt{3150}} = -.89$$

(To standardize a variable we subtract its expected value and divide this difference by the "standard deviation," which is another name for the square root of the variance.)

AREAS UNDER THE STANDARD NORMAL CURVE

The shaded area is tabulated for values of z from 0 to 3.08, in increments of .02.



z	.00	.02	.04	.06	.08
0.00	.0000	.0080	.0160	.0239	.0319
0.10	.0398	.0478	.0557	.0636	.0714
0.20	.0793	.0871	.0948	.1026	.1103
0.30	.1179	.1255	.1331	.1406	.1480
0.40	.1554	.1628	.1700	.1772	.1844
0.50	.1915	.1985	.2054	.2123	.2190
0.60	.2258	.2324	.2389	.2454	.2518
0.70	.2580	.2642	.2704	.2764	.2823
0.80	.2882	.2939	.2996	.3051	.3106
0.90	.3160	.3212	.3264	.3315	.3365
1.00	.3414	.3462	.3508	.3554	.3599
1.10	.3644	.3687	.3729	.3770	.3810
1.20	.3850	.3888	.3925	.3962	.3998
1.30	.4032	.4066	.4099	.4131	.4162
1.40	.4193	.4222	.4251	.4279	.4306
1.50	.4332	.4358	.4382	.4406	.4430
1.60	.4452	.4474	.4495	.4516	.4535
1.70	.4554	.4573	.4591	.4608	.4625
1.80	.4641	.4656	.4671	.4685	.4699
1.90	.4713	.4725	.4738	.4750	.4761
2.00	.4772	.4783	.4793	.4802	.4812
2.10	.4821	.4829	.4838	.4845	.4853
2.20	.4860	.4867	.4874	.4880	.4886
2.30	.4892	.4897	.4903	.4908	.4912
2.40	.4917	.4921	.4925	.4929	.4933
2.50	.4937	.4940	.4943	.4946	.4949
2.60	.4952	.4955	.4957	.4960	.4962
2.70	.4964	.4966	.4968	.4970	.4971
2.80	.4973	.4975	.4976	.4977	.4979
2.90	.4980	.4981	.4982	.4983	.4984
3.00	.4985	.4986	.4986	.4987	.4988

Since our table has no negative z-values it may seem that we are helpless. However, we can take advantage of the symmetry of the normal curve and determine the area (or probability) corresponding to values of z greater than .89 instead of the area to the left of −.89. We do this by subtracting the tabulated value .3133, corresponding to z = .89, from .5000, which is the total area underneath the normal curve to the right of zero. The result, .5000 − .3133 = .1867, or about 19%, is the chance we'd still be behind after making 2500 of these 2% favorable bets.

Distribution of a Point Count

We can also use the normal distribution to indicate how often different counts will occur for a point count system, providing that the number of cards left in the deck is specified. The following procedure can be used.

1. Calculate the sum of squares of the point values assigned to the thirteen denominations, calling it ss.

2. Calculate $b = \sqrt{\dfrac{ss\,(N-n)n}{13\ (N-1)}}$ where N is the number of cards in the full deck and n is the number of cards remaining.

3. Divide b into one half less than the count value you're interested in.

4. Divide b into one half more than the count value you're interested in.

5. The difference between the normal curve areas corresponding to the two numbers calculated in steps 3 and 4 will be the probability that the particular count value will occur.

As an example, suppose we wish to know the probability that there will be a +3 Hi Opt II count when there are 13 cards left from a single deck.

1. The point values for Hi Opt II are (0 1 1 2 2 1 1 0 0 −2), so $ss = 4(1)^2 + 2(2)^2 + 4(-2)^2 = 28$

2. N = 52 for a single deck and n = 13

$$b = \sqrt{\frac{28\ (52-13)13}{13\ (52-1)}} = 4.63$$

3. $3 - .5 = 2.5$; $z = 2.5/4.63 = .54$

4. $3 + .5 = 3.5$; $z = 3.5/4.63 = .76$

5. The area corresponding to .54 is .2054

   The area corresponding to .76 is .2764

The difference between these areas, and our answer, is .0710. The precise probability can be found in Appendix A of Chapter Seven, and is .069 to three digits after the decimal point.

How often is Strategy changed?

Although our only practical interest is in how much can be gained by varying basic strategy, we can also use the normal probability tables to estimate how often it should be done. To do so is quite simple. Going back to the example illustrating how much is gained by taking insurance with 40 cards left out of 51 (a single deck corrected for the dealer's up card), we need only the value $z = .57$. Then we subtract the area given in our normal curve probability charts corresponding to .57 from .5000. The result is $.5000 - .2157 = .2843$, and we estimate that insurance would be taken 28% of the time that the dealer has an ace showing with 40 cards remaining. (Precise calculations show the answer to be 25%.)

Similarly, we find the approximate probability of a favorable hit of hard 12 against the dealer's 6 with five cards left in the deck to be $.5000 - .0160 = .4840$, where the .0160 is the entry in our normal curve table corresponding to a z-value of .04. The exact probability is found in Chapter Eleven, and is .548.

Gain from Bet Variation

One of the most important uses to which the UNLLI can be put is measuring how much can be gained by betting one extra unit when the deck becomes favorable. To illustrate this, assume single deck play in Reno at a full table, so the player gets only one opportunity to raise his bet. $N = 52$ and we can take $n = 30$ since seven players, along with the dealer, would use up about $N - n = 22$ cards on the first round. From page 71, $m = -.45$ and $ss = 2.65$. Following the steps on page 88, we have:

1. $b = 2.76$
2. $z = .16$
3. From the UNLLI chart take .3240
4. 2.76 (.3240) = .89(%)

Thus we see that the basic strategist who diagnoses his advantage perfectly and bets k units when he perceives an advantage will make .89 (k—1)—.45 percent of a unit on the hand dealt with 30 cards left.

When the player has a basic strategy advantage for the full deck, then this computational technique can be used to measure how much will be saved by each extra unit which is *not* bet in *unfavorable* situations. In Chapter Eight we deduce that Atlantic City's six deck game with early surrender gave the basic strategist about a .17% edge. Using $m = .17$, $ss = 4.53$, $N = 312$, $n = 156$, and $\rho = .90$, we obtain:

1. $b = 1.71$, $\rho\, b = 1.54$
2. $z = .17/1.54 = .11$
3. .3464 from the UNLLI chart
4. 1.54 (.3464) = .53 (%), which is the amount gained on each unit *not* bet at the 156 card level by a player using a system with .90 betting correlation to decide when to *reduce* his wager.

APPENDIX TO CHAPTER 6

A.

The strategy tables presented are not the very best we could come up with in a particular situation. As mentioned in this chapter more accuracy can be obtained with the normal approximation if we work with a 51 rather than a 52 card deck. One could even have separate tables of effects for different two card player hands, such as (T,6) v T. Obviously a compromise must be reached, and my motivation has been in the direction of simplicity of exposition and ready applicability to multiple deck play.

B.

More precisely, the effect of removing a card from two decks is 51/103 of the single deck effect and for four decks it is 51/207 rather than 1/4.

C.

More accuracy is possible in the calculation of the Hi Lo system's single deck correlation with the insurance payoffs:

$$\rho = \frac{\sum_{1}^{n=51} x_i y_i - n\bar{x}\bar{y}}{\sqrt{\left(\sum x_i^2 - n\bar{x}^2\right) \cdot \left(\sum y_i^2 - n\bar{y}^2\right)}} =$$

$$\frac{16(2)(1)+3(-1)(1)+20(-1)(-1)-51\cdot\frac{-3}{51}\cdot\frac{-1}{51}}{\sqrt{\left(99-51\left(\frac{-3}{51}\right)^2\right)\cdot\left(39-51\left(\frac{-1}{51}\right)^2\right)}} = .79$$

To avoid the confusion engendered by point counts whose 51 values don't sum to zero, I would just as soon forego this slight improvement in accuracy achieved by removal of the dealer's up card.

7

ON THE LIKELY CONSEQUENCES OF ERRORS IN CARD COUNTING SYSTEMS

"A little learning is a dang'rous thing;
Drink deep, or taste not the Pierian spring:
There shallow draughts intoxicate the brain,
And drinking largely sobers us again."
—Alexander Pope

Suppose you're playing in a casino that uses all 52 cards, as the Nevada Club in Reno used to. You have the usual 16 against the dealer's ubiquitous Ten. We consider three different sets of remaining cards.

Unplayed Residue	Favorability of Hitting over Standing
4,T	-50%
4,4,T,T	0%
4,4,4,T,T,T	+10%

From this simple example follow two interesting conclusions:

1. Strategic favorabilities depend not strictly on the proportion of different cards in the deck, but really on the absolute numbers.

2. Every card counting system ever created would misplay at least one of these situations because the value of the card counting parameter would be the same in each case.

The mathematical analysis of blackjack strategies is only in rare instances what might be called an "exact science." Some questions, particularly those related to the insurance bet, can be easily answered with complete certainty by direct and exhaustive probability calculations. In theory all questions can be so addressed but in practice the required computer time is prohibitive.

We have already, to a reasonable degree, quantified the worth of different systems when played in the error free, transistorized atmosphere of the computer, devoid of the drift of cigar smoke, effects of alcohol, and distracting blandishments of the cocktail waitress. But what of these real battlefield conditions? To err is human and neither the pit boss, the dealer, nor the cards are divine enough to forgive.

Two Types of Error

There are two principal types of error in employing a count strategy: (1) an incorrect measure of the actual parameter which may be due to either an arithmetic error in keeping the running count or an inaccurate assessment of the number of cards remaining in the deck, and (2) an imprecise knowledge of the proper critical index for changing strategy.

It is beyond my scope to comment on the likelihood of numerical or mnemonic errors other than to suggest they probably occur far more often than people believe, particularly with the more complex point counts. It strikes me as difficult, for instance, to treat a seven as 7 for evaluating my hand, but as +1 for altering my running count and calling a five 5 for the hand and +4 for the count. The beauty of simple values like plus one, minus one, and zero is that they amount to mere recognition or non-recognition of cards, with *counting* (forward or backward), rather than *arithmetic* to continuously monitor the deck.

Commercial systems employing so called "true counts" (defined as the average number of points per card multiplied by 52) produce both types of error. The estimation of the number of half decks or quarterdecks remaining is necessarily imprecise: with 20 cards left unseen some might divide by 1/2, others by 1/4. The relative error in the presumed parameter could thus be six or seven twentieths, 30 or 35%. Published strategic indices themselves have usually been rounded to the nearest whole number, so a "true count" full deck parameter of

5 might have as much as a 10% error in it. It is the view of the salesmen of such systems that these errors are not serious; it is my suggestion that they probably are.

An Exercise in Futility

Even if the correct average number of points in the deck is available, there are theoretical problems in determining critical indices. When I started to play I faithfully committed to memory all of the change of strategy parameters for the Hi Lo system. It was not until some years later that I realized that several of them had been erroneously calculated.

For some time, I was firmly convinced that I should stand with 16 v 7 when the average number of points remaining equalled or exceeded .10. I now know the proper index should be .20. What do you think the consequences of such misinformation would be in this situation? Relatively minor? Not only was I playing the hand worse than a basic strategist, but, with 20 cards left in the deck I would have lost three times as much, at the 30 card level twenty times as much, and at the 40 card level five hundred times as much as knowledge of the correct parameter could have gained me.

The computer technique of altering normal decks so as to produce rich or lean mixtures for investigating different situations has not always incorporated an accurate alteration of conditional probabilities corresponding to the extreme values of the parameter assumed. The proper approach can be derived from bivariate normal assumptions and consists of maintaining the usual density for zero valued cards and displacing the other denominations in proportion to their assigned point values, rather than just their algebraic signs. [A]

As an example of the technical difficulties still to be encountered consider a +8/26 Hi Opt I deck. Computer averaging of all possible decks with this count leaves us with a not surprising "ideal" deck of twelve tens, one each three, four, five, and six, and two of everything else. It is by no means likely, however, that the favorabilities for this "ideal" deck will be precisely the average of those from all possible +8/26 decks (of which the non-ideal far outnumber the ideal). It would, for instance, be impossible to be dealt a pair of threes from such an ideal deck; a more reasonable estimate of the probability of this is 1/26 x 3/25 x 3/15, but even this is imprecise in the 3/25 which complete analysis shows to be 3.17828/25. There is at present no completely satisfactory resolution of such quandaries and even the most carefully computerized critical indices have an element of faith in them.

Behavior of Strategic Expectation as the Parameter changes

The assumption that the favorability for a particular action is a linear function of the average number of points in the deck is applied to interpolate critical indices and is also a consequence of the bivariate normal model used to analyze efficiency in terms of correlation coefficients. How valid is this assumption?

The answer varies, depending on the particular strategic situation considered. Tables 1 and 2, which present favorabilities for doubling down over drawing with totals of 10 and 11 and hitting over standing for 12 through 16, were prepared by using infinite deck analyses of the Hi Opt I and Ten Count strategies. Critical points interpolated from them should be quite accurate for multiple deck play and incorporating the effect of removing the dealer's up card permits the adjustment of expectations and indices for a single deck.

The most marked non-linearities are found when the dealer has a 9 or T showing. This is probably attributable to the fact that the dealer's chance of breaking such a card decreases very rapidly as the deck gets rich in tens. Linearity when the dealer shows an ace (dealer hits soft 17) is much better because player's and dealer's chance of busting grow apace.

To estimate how much conditional improvement the Hi Opt provides with 20 cards remaining in the deck multiply the Table 1 entries in the second through fifth columns by .22, .11, .03, and .01 respectively *if* they indicate a change in basic strategy. For the Ten Count multiply the Table 2 entries corresponding to ten densities of 9/26, 10/26, and 12/26 by .18, .10, .07, and .06, again only if they indicate a departure from basic play. For ten poor decks, multiply the 7/26, 6/26, 5/26, and 4/26 entries by .14, .11, .08, and .05. You will observe that many of the albeit technically correct parameters players *memorize are* virtually worthless.

TABLE 1

STRATEGIC FAVORABILITIES (IN %) AS A FUNCTION OF HI OPT PARAMETER

(Hi Opt parameter quoted is average number of points in deck. Point values for this system are +1 for 3,4,5,6 and -1 for Tens)

DEALER'S CARD 2

0/10	-1/10	-2/10	-3/10	-4/10	TOTAL
17.6	7.4	- 6.3	-21.9	-38.9	10
23.3	12.2	- 2.2	-18.3	-35.7	11
3.3	11.1	18.2	24.6	30.3	12
- 2.2	7.1	15.7	23.7	31.1	13
- 7.7	3.4	13.9	23.9	33.5	14
-13.2	- .2	12.4	24.5	36.4	15
-18.7	- 6.5	5.3	16.9	28.2	16

DEALER'S CARD 3

0/10	-1/10	-2/10	-3/10	-4/10	TOTAL
20.3	9.8	- 4	-20.2	-37.7	10
25.8	14.5	- .1	-16.7	-34.6	11
1.3	9.8	17.3	24.1	30	12
- 4.5	5.5	14.6	23	30.8	13
-10.3	1.5	12.6	23.1	33.1	14
-16.1	- 2.3	10.9	23.6	35.9	15
-21.9	- 8.9	3.7	15.8	27.6	16

DEALER'S CARD 4

0/10	-1/10	-2/10	-3/10	-4/10	TOTAL
23	12.1	- 2	-18.7	-36.7	10
28.3	16.7	2	-15.2	-33.5	11
- .8	8.3	16.3	23.3	29.5	12
- 6.9	3.7	13.4	22.1	30.1	13
-13	- .5	11.1	22	32.2	14
-19.1	- 4.6	9.2	22.3	34.9	15
-25.2	-11.4	1.8	14.3	26.5	16

DEALER'S CARD 5

0/10	-1/10	-2/10	-3/10	-4/10	TOTAL
25.6	14.5	.7	-16.1	-34.4	10
30.7	19	4.7	-12.7	-31.3	11
- 2.9	6.2	14.3	21.5	28.1	12
- 9.3	1.3	11	20	28.4	13
-15.7	- 3.3	8.4	19.5	30.2	14
-22.1	- 7.7	6.1	19.5	32.6	15
-28.6	-14.8	- 1.5	11.3	24.1	16

DEALER'S CARD 6

0/10	-1/10	-2/10	-3/10	-4/10	TOTAL
28.2	18	5.5	-10.7	-29.5	10
33.2	22.4	9.3	- 7.4	-26.6	11
- 5.1	3.6	11.7	19.2	26.2	12
-11.9	- 1.7	7.9	17.1	25.9	13
-18.7	- 6.8	4.8	16	27.2	14
-25.4	-11.6	1.9	15.4	29	15
-32.2	-19.2	-6.3	6.7	20	16

DEALER'S CARD 7

0/10	1/10	2/10	3/10	4/10	TOTAL
18.2	14.1	10.2	6.4	2.7	13
14.1	8.5	3	- 2.5	- 8	14
10.1	3.2	- 3.4	- 9.9	-16.4	15
6.1	1.1	- 3.5	- 7.8	-11.8	16

0/10	-1/10	-2/10	-3/10	-4/10	TOTAL
14.2	3	- 9.1	-22.4	-37.3	10
17.7	6.8	- 5.2	-18.6	-33.7	11

DEALER'S CARD 8

0/10	1/10	2/10	3/10	4/10	TOTAL
12.7	7.7	3.4	- .4	- 3.6	14
9	2.4	- 3.6	- 9	-14	15
5.2	.1	- 4.3	- 8.1	-11.4	16

0/10	-1/10	-2/10	-3/10	-4/10	TOTAL
9.4	- .7	-12	-24.9	-39.5	10
12.6	2.5	- 8.8	-21.6	-36.2	11

DEALER'S CARD 9

0/10	1/10	2/10	3/10	4/10	TOTAL
10.4	6.2	3	.9	0	14
6.9	1.1	- 3.7	- 7.5	-10.1	15
3.4	- 1.3	- 5	- 7.8	- 9.6	16

0/10	-1/10	-2/10	-3/10	-4/10	TOTAL
3.2	- 5.4	-15.5	-27.4	-41.3	10
7.4	- 1.5	-11.9	-23.9	-38	11

DEALER'S CARD T

0/10	1/10	2/10	3/10	4/10	TOTAL
- 3.2	.3	2.5	3.7	4.2	10
6.2	12.6	18.2	23.3	28.3	11
14.2	11.9	10.5	10.4	11.5	12
10.7	7.8	6.2	5.8	6.9	13
7.1	3.1	.4	- 1	- .9	14
3.6	- 2	- 6.4	- 9.4	-11.1	15
.1	- 4.4	- 7.6	- 9.7	-10.5	16

0/10	-1/10	-2/10	-3/10	-4/10	TOTAL
6.2	- 1.8	-11.2	-22.3	-35.4	11

DEALER'S CARD A

0/10	1/10	2/10	3/10	4/10	TOTAL
- 6.4	5.6	15.2	23.3	29.8	10
.6	13.5	24.2	33.5	41.4	11
18	13.2	8.2	3.1	- 1.5	12
14.9	8.8	2.5	- 3.6	- 9.2	13
11.8	3.9	- 4.1	-11.9	-19	14
8.7	- 1.2	-11.2	-20.9	-29.8	15
5.7	- 3.7	-12.9	-21.7	-29.7	16

0/10	-1/10	-2/10	-3/10	-4/10	TOTAL
.6	-13.8	-29.3	-45.8	-63	11

TABLE 2

STRATEGIC FAVORABILITIES (IN %) AS A FUNCTION OF TEN DENSITY

(Ten count parameter quoted is the fraction of tens in the deck; 8/26 is normal.)

DEALER'S CARD 2

8/26	7/26	6/26	5/26	4/26	TOTAL
17.6	10.9	3.4	- 4.9	-13.8	10
23.3	15.7	7.2	- 2.2	-12.2	11
3.3	9.9	16.2	22.2	27.8	12
- 2.2	4.5	11	17.1	22.9	13
- 7.7	- .8	5.7	12	17.9	14
-13.2	- 6.2	.5	6.9	13	15
-18.7	-11.6	- 4.8	1.8	8.1	16

DEALER'S CARD 3

8/26	7/26	6/26	5/26	4/26	TOTAL
20.3	13.8	6.4	- 1.8	-11	10
25.8	18.4	10.1	.9	- 9.4	11
1.3	8.1	14.5	20.7	26.6	12
- 4.5	2.4	9	15.3	21.4	13
-10.3	- 3.3	3.4	9.9	16.2	14
-16.1	- 9	- 2.1	4.5	11	15
-21.9	-14.7	- 7.7	- .9	5.8	16

DEALER'S CARD 4

8/26	7/26	6/26	5/26	4/26	TOTAL
23	16.7	9.6	1.6	- 7.6	10
28.3	21.2	13.2	4.2	- 6	11
- .8	6.1	12.8	19.1	25.2	12
- 6.9	.1	6.9	13.4	19.7	13
-13	- 5.9	1	7.6	14.1	14
-19.1	-11.9	- 4.9	1.9	8.6	15
-25.2	-18	-10.8	- 3.8	3.1	16

DEALER'S CARD 5

8/26	7/26	6/26	5/26	4/26	TOTAL
25.6	19.6	12.8	5.1	- 3.7	10
30.7	24	16.3	7.6	- 2.2	11
- 2.9	4.1	10.9	17.4	23.6	12
- 9.3	- 2.2	4.6	11.3	17.7	13
-15.7	- 8.6	- 1.6	5.2	11.8	14
-22.1	-14.9	- 7.9	- .9	5.9	15
-28.6	-21.3	-14.1	- 7	.1	16

DEALER'S CARD 6

8/26	7/26	6/26	5/26	4/26	TOTAL
28.2	22.6	16.2	8.8	.4	10
33.2	26.9	19.6	11.3	1.9	11
- 5.1	1.9	8.7	15.3	21.6	12
-11.9	- 4.8	2.1	8.8	15.3	13
-18.7	-11.5	- 4.6	2.3	9	14
-25.4	-18.3	-11.2	- 4.2	2.7	15
-32.2	-25	-17.8	-10.7	- 3.6	16

DEALER'S CARD 7

8/26	7/26	6/26	5/26	4/26	TOTAL
14.2	8.8	3.1	- 3	- 9.6	10
17.7	11.8	5.6	- 1.2	- 8.4	11

DEALER'S CARD 8

8/26	9/26	10/26	11/26	12/26	TOTAL
12.7	10.1	7.7	5.5	3.6	14
9	6.6	4.5	2.6	.9	15
5.2	3.1	1.2	- .4	- 1.8	16

8/26	7/26	6/26	5/26	4/26	TOTAL
9.4	4	- 1.7	- 7.8	-14.2	10
12.6	6.8	.6	- 6.1	-13.2	11

DEALER'S CARD 9

8/26	9/26	10/26	11/26	12/26	TOTAL
17.4	14	10.8	7.8	5.1	12
13.9	10.7	7.8	5	2.6	13
10.4	7.4	4.7	2.3	.1	14
6.9	4.2	1.7	- .5	- 2.5	15
3.4	.9	- 1.4	- 3.3	- 5	16

8/26	7/26	6/26	5/26	4/26	TOTAL
3.2	- 2.1	- 7.7	-13.5	-19.7	10
7.4	1.5	- 4.7	-11.3	-18.3	11

DEALER'S CARD T

8/26	9/26	10/26	11/26	12/26	TOTAL
- 3.2	- .7	1.1	2.4	3.3	10
6.2	11.3	16	20.5	24.7	11
14.2	10.3	6.8	3.5	.5	12
10.7	7	3.7	.7	- 2	13
7.1	3.7	.7	- 2.1	- 4.5	14
3.6	.4	- 2.4	- 4.9	- 7	15
.1	- 2.9	- 5.4	- 7.7	- 9.5	16

8/26	7/26	6/26	5/26	4/26	TOTAL
6.2	.7	- 5.2	-11.5	-18.2	11

DEALER'S CARD A

8/26	9/26	10/26	11/26	12/26	TOTAL
- 6.4	1.8	9.2	15.9	21.8	10
.6	9.7	18.1	25.6	32.3	11
18	13.2	8.1	2.7	- 2.9	12
14.9	10	4.9	- .5	- 6.1	13
11.8	6.8	1.6	- 3.8	- 9.4	14
8.7	3.7	- 1.6	- 7	-12.7	15
5.7	.5	- 4.8	-10.3	-15.9	16

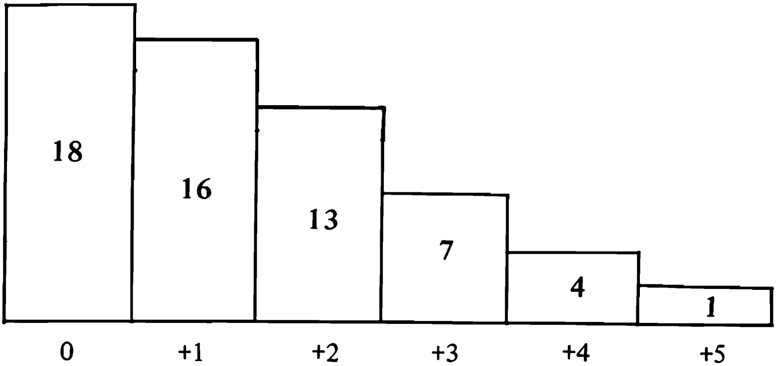
8/26	7/26	6/26	5/26	4/26	TOTAL
.6	- 8.8	-18.5	-28.5	-38.7	11

The case of 14 against dealer's ten provides an interesting exercise in futility: having paid a bundle for the technically correct Hi Opt parameter and committed it to memory, how much is this information worth? Assuming 20 cards left in the deck and that the player holds 14 against a ten, he will gain .036% above the basic strategy by perfect employment of the sacred index. A superstitious player who only counts sevens and stands when all of them are gone will gain 1.4% under the same conditions, an almost forty fold improvement!

An Explanation of Errors

Even if not always realized in practice, the linear assumption that the player's conditional gain or loss is a constant times the difference between the proper critical index and the current value of his parameter provides a valuable perspective to illustrate the likely consequences of card counting errors. Whatever their source (the type (1) and (2) errors mentioned earlier), the player will either be changing strategy too often, equivalent to believing the critical index is less extreme than it really is, or not changing strategy enough, equivalent to believing the critical index is more extreme than it actually is.

The subject can perhaps be demystified by appeal to a graphic. At a certain level of the deck the running count will tend to have a probability distribution like the one below, where the numbers inside the rectangles are the frequencies (in %) of the different count values. (Only the positive half of the distribution is shown.)



Suppose, now, that the threshold change in strategy parameter is +2. This means that there will be neither gain nor loss from changing strategy for a running count of +2, but there will be a conditional loss at any count less than +2 and a conditional gain at any count greater than +2. The much bandied "assumption of linearity" means that the gain or loss will be precisely proportional to the distance of the actual running count from the critical count of +2.

Now suppose one was (for whatever reason) addicted to premature changing of strategy for counts of +1 or higher. He would realize an overall profit of $16(-1) + 13(0) + 7(+1) + 4(+2) + 1(+3) = 2$. The supercautious player who only changed strategy with a count of +5 would do better, realizing $1(+3) = 3$ units even though in a sense his belief about when to change strategy was farther from the truth.

What we see, of course, is that counts closer to zero (like +1) are much more likely to occur than the more extreme ones where most of the conditional profit lies. To fix the idea in your mind try to show, using the diagram, that if the critical threshold value is +3, the player who changes strategy for +2 or above will lose more than the basic strategist (who never changes), and also will lose more than the perfect employer of the system can gain.[B]

Overall it seems, then, that the consequences of changing strategy too frequently will be more serious than those of not changing strategy often enough. Indeed, the Baldwin group foresaw this in their book: "Ill considered changes will probably do more harm than good. . .Many players overemphasize the last few draws and, as a result, make drastic and costly changes in their strategy."

This will be particularly true if the actual blackjack situation's departure from linearity is such that the rate of change of favorability falls off with increasing parameter values; this is quite characteristic of standing with stiff hands against 8, 9, or ten. This suggests that it would be a service to both the memory and pocket book to round playing indices to the nearest conveniently remembered and more extreme value.

That players, especially the mathematically inclined who are fascinated by such puzzles, may tend to become over enamored of the possibilities of varying basic strategy is the experience of the author. There is, as in poker, a tendency to "fall in love with one's cards"[1] which may cause pathologists to linger over unfavorable decks (where much of this action is found) for the sole purpose of celebrating their knowledge with a bizarre and eye-opening departure play. This is an understandable concomitant of the characteristic which best differentiates the casino blackjack player from the independent trials gambler, namely a desire to exercise control over his own destiny.

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[1]Cutler, W. H. *An Optimal Strategy for Pot Limit Poker.* The American Mathematical Monthly, Vol. 82, No. 4, April, 1975.

APPENDIX TO CHAPTER 7

A.

The instantaneous value of any point count system (whether it uses + or − 0, 1, 2, 3, 4, 7, 11 etc.) induces a certain conditional probability distribution for the remaining cards. It has already been shown in Chapter Five that cards assigned the value of zero are uncorrelated with the system's parameter and hence tend to have the same neutral distribution regardless of the sign or magnitude of the point count. We shall now show that more generally, as the count fluctuates, we are entitled to presume a deflection in a card denomination's density proportional to the point value assigned to it.

Towards this end we again consider the +1, −12 indicator count for a particular denomination. If P_i, i = 1 to 13, are the system's point values, the correlation between the indicator of the k^{th} denomination and the original system will be

$$\frac{13\,P_k}{\sqrt{156 \cdot \sum P_i^2}}$$

and hence proportional to P_k itself.

Our demonstration is concluded by observing that the deflection of the conditional mean of the indicator count from its overall mean will be proportional to this correlation, and hence proportional to P_k, as promised.

The following tables illustrate this phenomenon by cataloging the average deflection from normal of the differently valued denominations of the Hi Opt II system (0 1 1 2 2 1 1 0 0 -2) for various positive counts with 13, 26 and 39 cards left in the deck. (The deflections for negative counts with 39, 26, and 13 cards remaining can be obtained by merely changing the algebraic signs in the 13, 26, and 39 card positive count tables.)

TABLE 3

DEFLECTION FROM NORMAL OF HI OPT II CARDS

A = Running Count
B = Deflection of Cards Valued +2 (Counted as -2)
C = Deflection of 0 Valued Cards
D = Deflection of Cards Valued -1 (Counted as +1)
E = Deflection of -2 Valued Cards (Counted as +2)
F = Probability the Particular Count Value Will Occur

13 Cards Left

A	B	C	D	E	F
0	-0.009	0.020	0.007	-0.026	.086
1	0.063	0.019	-0.029	-0.095	.083
2	0.136	0.015	-0.066	-0.163	.078
3	0.209	0.010	-0.103	-0.228	.069
4	0.283	0.005	-0.141	-0.293	.059
5	0.359	-0.003	-0.180	-0.353	.048
6	0.435	-0.014	-0.218	-0.412	.038
7	0.511	-0.024	-0.258	-0.470	.028
8	0.589	-0.036	-0.300	-0.523	.020
9	0.668	-0.055	-0.340	-0.574	.014
10	0.747	-0.070	-0.383	-0.623	.009
11	0.826	-0.087	-0.424	-0.674	.005
12	0.909	-0.111	-0.471	-0.711	.003
13	0.993	-0.141	-0.510	-0.755	.002
14	1.071	-0.154	-0.554	-0.803	.001
15	1.162	-0.190	-0.612	-0.814	.000

26 Cards Left

0	0.000	0.000	0.000	0.000	.074
1	0.071	0.000	–0.035	–0.072	.073
2	0.143	–0.001	–0.070	–0.144	.069
3	0.214	–0.001	–0.106	–0.216	.064
4	0.286	–0.001	–0.141	–0.288	.056
5	0.357	–0.001	–0.177	–0.359	.048
6	0.428	–0.001	–0.212	–0.431	.040
7	0.500	–0.001	–0.248	–0.502	.032
8	0.571	–0.001	–0.285	–0.573	.025
9	0.643	0.000	–0.321	–0.643	.018
10	0.714	0.000	–0.358	–0.713	.013
11	0.786	0.001	–0.395	–0.783	.009
12	0.857	0.002	–0.433	–0.852	.006
13	0.929	0.004	–0.472	–0.920	.004
14	1.001	0.006	–0.511	–0.988	.002
15	1.072	0.008	–0.551	–1.055	.001
16	1.144	0.011	–0.591	–1.122	.001
17	1.215	0.015	–0.633	–1.187	.000

39 Cards Left

0	0.009	–0.020	–0.007	0.026	.085
1	0.080	–0.019	–0.042	–0.048	.083
2	0.151	–0.017	–0.077	–0.122	.078
3	0.220	–0.014	–0.111	–0.198	.070
4	0.289	–0.007	–0.144	–0.278	.060
5	0.356	0.002	–0.177	–0.360	.049
6	0.423	0.011	–0.209	–0.444	.038
7	0.489	0.024	–0.241	–0.531	.028
8	0.553	0.041	–0.272	–0.623	.020
9	0.615	0.060	–0.301	–0.718	.013
10	0.677	0.082	–0.331	–0.816	.008
11	0.736	0.110	–0.359	–0.919	.005
12	0.792	0.146	–0.386	–1.031	.003
13	0.845	0.188	–0.410	–1.150	.001
14	0.895	0.235	–0.434	–1.275	.001
15	0.940	0.297	–0.455	–1.415	.000

As a specific example, the table informs us that with a +3 count and 13 cards remaining, we expect 1.209 Tens, Jacks, Queens, and Kings, 1.010 Aces, Nines, and Eights, .897 Twos, Threes, Sixes, and Sevens, and .772 Fours and Fives.

Observe that the B and E columns tend to be close in magnitude, but opposite in sign, the C column is generally close to zero, and the D column is about half of E. This is what the ideal theory suggests will happen.

B.

Table 4 was prepared by a probabilistic analysis of Hi Opt I parameters with 20, 30, and 40 cards left in a single deck. The lessons to be learned from it would seem to apply to any count system. Examined critical indices range from .06 to .24 and the changes in favorability are assumed to be linear in the previously described sense. The body of the table quantifies the player's cumulative gain or loss from changing strategy with possible "action indices" as, or more, extreme than those which appear in the left hand margin. The units are arbitrarily scaled to avoid decimals; they would actually depend on the volatility of, and point count's correlation with, the particular situation considered.

For relatively small critical indices such as .06 there isn't much danger from premature change of strategy since there is little opportunity to go wrong (it is assumed the player never makes a mistake on the wrong side of zero). However for larger critical indices the player may lose more from such overzealousness than someone else playing the system correctly can gain. For example, it would seem innocuous to mistake a critical index of .24 for one of .20, but the table shows that with 30 cards remaining it would cost the player 50 units, whereas perfect card counting can produce only 14 units.

This table can also be used to assess how well a "running count" strategy would fare relative to a strategy based on a "true" knowledge of the average number of points remaining in the deck. Imagine that the situation with critical index .12 is typical of variations in strategy overall. If opportunity arises three times, with 20, 30 and 40 cards remaining, the total

TABLE 4

ACTION INDEX	CRITICAL INDICES			
	.06	.12	.18	.24
20 CARDS				
.05	2926	353	−2220	−4793
.1	3060	1289	− 482	−2252
.15	2617	1511	404	− 702
.2	1890	1268	647	25
.25	1166	855	543	232
.3	616	478	341	203
.35	278	225	171	118
.4	106	89	71	53
.45	34	29	24	19
.5	9	8	7	5
30 CARDS				
.033	1144	−1435	−4014	−6594
.067	1496	− 291	−2079	−3867
.1	1423	295	− 834	−1962
.133	1100	456	− 187	− 830
.167	715	387	58	− 270
.2	396	247	98	− 50
.233	187	128	68	9
.267	75	55	34	14
.3	25	19	13	7
.333	7	6	4	3
40 CARDS				
.025	−133	−2642	−5151	−7661
.05	395	−1210	−2816	−4421
.075	512	− 387	−1286	−2185
.1	396	− 38	− 471	− 905
.125	225	48	− 129	− 306
.15	98	38	− 21	− 81
.175	33	16	0	− 16
.2	8	5	1	− 2
.225	1	1	0	0
.25	0	0	0	0

possible gain from perfect employment of the system is 1511 + 456 + 48 = 2015, the sum of the largest entry in each column.

A "running count" player, making no effort to adjust for depth in the deck, would gain less than this, depending on the critical running count he used. If he changed strategy for running counts of +2 or more he would be a net loser, by 1289 − 291 − 1210 = −212. Changing strategy for +4 or more will optimize "running count" gain at 1268 + 456 − 38 = 1686. This would seem to suggest that such a player would be able to pick up about 84% of the system's available gain (1686/2015),but this is to ignore that the deck would not always be dealt to the level assumed in our selection of the best running count value of +4. Furthermore, such numbers, already ingrained in the memory, would not be readily convertible for multiple deck play.

8

MANY DECKS AND DIFFERENT RULES

"...thinking as always that some idle instinct bet might carry the whole thing off. But no. Just another two bucks down the tube. You bastards."

—*Hunter Thompson,* **Fear and Loathing in Las Vegas**

With the rules generally presumed in this book, the player's expectations for basic strategy against one, four, and an infinite number of decks are +.02%, − .48%, and − .65% respectively. The chances of being dealt a blackjack are 128/2652 = .04827 for one deck, 128/2678 = .04779 for two decks, 128/2691 = .04757 for four decks, and 128/2704 = .04734 for an infinite deck.*

Probabilities and expectations associated with different numbers of decks obey a curious metric and appear to be predictable by interpolation using the reciprocal of the number of decks employed. Thus to estimate what expectation our rules would produce for a double deck we would pick −.31%, half way between the single deck figure of +.02% and the infinite deck −.65%, since 1/2 is half way between 1/1 and 1/∞. Likewise we could extrapolate a .02 + .67 = .69% advantage for a half deck, and this isn't far from Thorp's value of .85% when doubling after splitting is taken into account.[A,B]

What is it about the multiple deck which makes it inherently less favorable? To begin with, almost half of the .67% difference in expectation between one deck and many decks can be traced to the fact that the favorability of doubling down is reduced from 1.59% for one deck to 1.34% for the infinite deck. The double down pair often contains two cards the player does not wish to draw and their removal significantly improves the chance of a good hand from one deck but is negligible otherwise. A good example of this is doubling nine against

*See page 170 for explanation of infinite deck.

a deuce; our chance of drawing a ten (or the dealer having one underneath his 2) is 16/49 = .326 for one deck but only 64/205 = .312 for four decks, where the double is a very marginal play.

Of the unexplained .42% remaining difference in expectation, .07% can be attributed to the bonus paid the player for uncontested blackjacks, but the multiple deck player appears to make that up by more frequent pair splitting activity. Presumably the remaining discrepancy reflects the player's gain by judicious standing with stiff totals. For example, standing with (7,6) v 4 is about 12% better than hitting it in one deck, but less than an 8% improvement in the four deck game. A stiff hand usually contains at least one card, and often several, which would help the dealer's up cards of two through six, against which this option is exercised, and the favorable effect of their removal (i.e. their appearance in the player's hand) is dampened in the multiple deck game.

The Effect of Rule Changes

In the next table the effect of some rule changes occasionally encountered is given for both one deck and an infinite number of decks. The reader can use interpolation by the reciprocal of the number of decks to get an estimate of what the effects would be for two and four decks. For instance, if doubling soft hands is forbidden in a four deck game, take one fourth of the difference between the −.13% given for one deck and the −.08% for the infinite deck. This is .01%, and hence we presume a −.09% effect with four decks. Similarly, we get −.11%, half way between the two figures in the table, as the double deck penalty for prohibition of soft doubling.

Notice how splitting is more valuable for the infinite deck due to the greater likelihood of pairs being dealt. Doubling down after pair splitting is worth the same in each case because the reduced frequency of pairs in the single deck is nullified by the increased advantage on double downs.

CHANGES IN EXPECTATION DUE TO VARIATION IN RULES (IN %)

	One Deck	Infinite Deck
No doubling on 11	−.81	−.73
No doubling on 10	−.52	−.45
No doubling on 9	−.132	−.076
No doubling Soft Hands	−.131	−.083
No split of Non-Aces	−.21	−.25
No split of Aces	−.16	−.18
No resplit of Non-Aces	−.018	−.039
Resplit of Aces allowed	.03	.08
Double after split	.14	.14
Double after split, when no resplit	.13	.12
Double 11 after split	.07	.07
Double 10 after split	.05	.05
Double three or more cards	.24	.22
Two to one blackjack	2.32	2.25
Dealer hits soft 17	−.19	−.22

Opportunity arises slowly in Multiple Decks

From the card counter's viewpoint another important difference between one and many decks is the slowness with which the deck's original condition changes. Each row of the following table provides a comparison of the fluctuations in various numbers of decks by display of the number of remaining cards which would have the same degree of fluctuation associated. For example, the amount of opportunity likely to be encountered with 31 cards left in a single deck is equivalent to what would occur with 44 cards in two decks, 55 in four, 61 in six, and 75 left out of an infinite pack.[C]

One Deck	Two Decks	Four Decks	Six Decks	Infinite Deck
1	1	1	1	1
11	12	13	13	14
21	26	30	31	35
31	44	55	61	75
41	67	100	118	190
51	100	193	279	2601

The important thing to realize is how many cards must be removed from multiple decks before they become as interesting as a single deck. Seeing one card from a single deck entitles us to as much excitement as will glimpsing of $312 - 279 = 33$ cards from six decks. If you're playing at that great blackjack table in the sky (where St. Peter deals and you know who is the pit boss), you'll have to wait an eternity, or until 2601 cards are left, before the degree of departure from normal composition is equivalent to that produced by the observation of the burn card from a standard pack of 52.

We see that the last few cards of a multiple deck can be slightly more favorable for both betting and playing variations than the corresponding residue from a single deck. However, it must be kept in mind that such situations are averaged over the entire deck when assessing overall favorability. An interesting consequence of this is that even if one had the time to count down an infinite deck, it would do no good since the slightly spicier situations at the end would still average out to zero. When we recall that the basic multiple deck games are inherently less advantageous, the necessity of a very wide betting range must be recognized.

Absolute efficiencies of card counting systems will decrease mildly, perhaps by three per cent for four decks. Since this decrease will generally be uniform over most aspects of the game, relative standings of different systems should not differ appreciably from those quoted in Chapter Four.

Betting Gain in two and four Decks

At the beginning of Chapter Six there is a table of exact insurance expectations along with a rule of approximation for other strategy gains which gives an indication of the futility of trying to make a living by flat betting a four-deck shoe. The next table shows how much profit accrues from betting one extra unit in favorable situations for two and four deck games played according to the rules generally presumed in Chapter Two. It can be derived (as were the similar single deck figures on page 28) by the method outlined on page 88, with ss = 2.84 and values of m = −.31 and −.48.

GAIN PER HAND FROM BETTING
ONE EXTRA UNIT IN FAVORABLE
SITUATIONS (%)

Number of Cards Remaining	Double Deck	Four Decks
10	2.73	2.72
20	1.77	1.80
30	1.32	1.39
40	1.04	1.13
50	.83	.96
60	.66	.82
70	.51	.71
80	.37	.62
90	.24	.55
100	.07	.48
110		.42
120		.36
130		.31
140		.26
150		.22
160		.18
170		.13
180		.09
190		.05
200		.01

The table can provide us with an estimate of profit from betting k units on each favorable hand, one otherwise. If a four deck player's last hand is dealt with 60 cards left, we average all the gains (including the .00 for the first, or come out, hand) down to that level:

$$\frac{.00 + .01 + .05 + .09 + \cdots + .71 + .82}{16} = .32$$

Now, multiply .32 by (k −1) and subtract .48, the full deck disadvantage. This is the average profit per hand (in %). Notice that k = 2.5 is necessary just to break even in this case. Although we've neglected strategy variation this is partially compensated by the assumption that the player diagnoses his basic strategy advantage perfectly.

The rest of the chapter will be devoted to certain uncommon but interesting variations in rules. Since these usually occur in conjunction with four deck games, this will be assumed unless otherwise specified.

No hole Card

With "English rules" the dealer does not take a hole card, and in one version, the player who has doubled or split a pair loses the extra bet if the dealer has a blackjack. In such a case the player minimizes his losses by foregoing eight splitting and doubling on 11 against the dealer's ten and ace and also not splitting aces against an ace. The primary penalty paid is that the correct basic strategy is not used when the dealer doesn't have blackjack. This costs .10% when the dealer shows a ten and .01% for an ace up. In another version, though, the player's built up 21 is allowed to push the dealer's natural; this favors the player by .17% against a ten and also .17% against an ace.

Surrender

"Surrender" is another, more common, rule. With this option the player is allowed to give up half his bet without finishing the hand if he doesn't like his prospects. Usually this choice must be made before drawing any cards. Since the critical expectation for surrendering is −.500, the following tables of infinite deck player expectations for totals of 4 through 21 against up cards of ace through ten will be of interest. They will also be useful for discussion of subsequent rule variations.

PLAYER'S EXPECTATION

Total \ Up Card	A	2	3	4	5	6	7	8	9	T
4	-.253	-.115	-.083	-.049	-.012	.011	-.088	-.159	-.241	-.289
5	-.279	-.128	-.095	-.061	-.024	-.001	-.119	-.188	-.267	-.313
6	-.304	-.141	-.107	-.073	-.035	-.013	-.152	-.217	-.293	-.338
7	-.310	-.109	-.077	-.043	-.007	.029	-.069	-.211	-.285	-.319
8	-.197	-.022	.008	.039	.071	.115	.082	-.060	-.210	-.249
9	-.066	.074	.121	.182	.243	.317	.172	.098	-.052	-.153
10	.081	.359	.409	.461	.513	.576	.392	.287	.144	.025
11	.143	.471	.518	.566	.615	.667	.463	.351	.228	.180
12	-.351	-.253	-.234	-.211	-.167	-.154	-.213	-.272	-.340	-.381
13	-.397	-.293	-.252	-.211	-.167	-.154	-.269	-.324	-.387	-.425
14	-.440	-.293	-.252	-.211	-.167	-.154	-.321	-.372	-.431	-.466
15	-.480	-.293	-.252	-.211	-.167	-.154	-.370	-.417	-.472	-.504
16	-.517	-.293	-.252	-.211	-.167	-.154	-.415	-.458	-.509	-.540
17	-.478	-.153	-.117	-.081	-.045	.012	-.107	-.382	-.423	-.420
18	-.100	.122	.148	.176	.200	.283	.400	.106	-.183	-.178
19	.278	.386	.404	.423	.440	.496	.616	.594	.288	.063
20	.655	.640	.650	.661	.670	.704	.773	.792	.758	.555
21	.922	.882	.885	.889	.892	.903	.926	.931	.939	.963

As indicated, the player surrenders if his expectation is worse (more negative) than −.500. Thus surrendering 16 v T saves the player .04, or 4%, when it happens. Naturally, the precise saving depends on what cards the player holds and on how many decks are used, but these tables are quite reliable for four deck play.

Some casinos even allow "early surrender", before the dealer has checked his hole card for a blackjack. This is quite a picnic for the knowledgeable player, particularly against the dealer's ace. We must revise the previous table of expectations

PLAYER'S EXPECTATION
(Dealer hits soft 17)

Total \ Up Card	A	2	3	4	5	6
4	-.292	-.113	-.081	-.047	-.011	.026
5	-.316	-.126	-.093	-.059	-.023	.015
6	-.341	-.138	-.105	-.070	-.034	.005
7	-.349	-.110	-.077	-.043	-.007	.030
8	-.263	-.025	.006	.037	.070	.104
9	-.124	.072	.119	.180	.242	.305
10	.033	.357	.407	.459	.512	.565
11	.103	.470	.517	.566	.614	.665
12	-.384	-.254	-.234	-.206	-.165	-.121
13	-.428	-.287	-.247	-.206	-.165	-.121
14	-.469	-.287	-.247	-.206	-.165	-.121
15	-.507	-.287	-.247	-.206	-.165	-.121
16	-.542	-.287	-.247	-.206	-.165	-.121
17	-.516	-.156	-.120	-.083	-.046	-.006
18	-.226	.110	.138	.166	.195	.223
19	.188	.378	.397	.416	.436	.453
20	.602	.635	.646	.657	.668	.678
21	.904	.880	.884	.887	.891	.894

(which are conditional on the dealer not having a blackjack), before determining the critical expectation for early surrender. In the infinite deck game this is done by solving the equations

$\frac{9}{13}E - \frac{4}{13} = -\frac{1}{2}$ and $\frac{12}{13}E - \frac{1}{13} = -\frac{1}{2}$ for the ace and ten respectively. This gives $E = -\frac{5}{18} = -.278$ as the critical point for surrendering against an ace and $E = -\frac{11}{24} = -.458$ against a ten. Looking back at the first table we perceive a marginally favorable early surrender with hard five against an ace and hard fourteen against a ten.[D]

To assess the full value of surrender to the player, account must be taken of the frequency of initial player hard totals and dealer's up cards. When this is done we get the following table of gain from proper strategy.

SURRENDER GAIN (IN %)

	Ace Up	Ten Up	9 Up	Total	Number of Decks
Conventional	.006(.028)	.075	.004	.085(.107)	Infinite
Conventional	.005(.024)	.063	.001	.069(.088)	Four
Conventional	.002(.017)	.020		.022(.036)	One
Early	.38 (.47)	.25	.004	.63 (.72)	Infinite
Early	.39 (.48)	.23	.001	.62 (.71)	Four
Early	.43 (.51)	.19		.62 (.70)	One

(Figures in parentheses indicate gain when dealer hits soft 17.)

When surrender is allowed at any time, and not just on the first two cards, the rule will be worth almost twice as much for conventional surrender and either 10% or 50% more for early surrender depending on whether the dealer shows an ace or a ten.

Bonus for multicard Hands

If the Plaza in downtown Las Vegas had had the "Six Card Automatic Winner" rule, I would have been spared the disappointment of losing with an eight card 20 to the dealer's three card 21. Six card hands are not very frequent and the rule is worth about .10% in a single deck and .15% for four decks. The expectation tables suggest a revised five card hitting strategy to cope with the rule in four decks: hit hard 17 v 9, T, and A; hit hard 16 and below v 2 and 3; hit hard 15 and below v 4, 5, and 6.

Some Far Eastern casinos have a sort of reverse surrender rule called "Five Card," wherein the player may elect to turn in any five card hand for a payment (to him) of half his bet. Again the table of expectations comes in handy, both for decisions on which five card hands to turn in and also for revision of four card hitting strategies.

A five card hand should be turned in if its expectation is less than +.500 and the difference is what the player gains.

Turning in a five card 16 against a dealer Ten is worth .500 − (−.540) = 1.040, or 104% of the original bet. A revised and abbreviated four card strategy is as follows:

Hit Soft 19 and Below Againsι Anything But a 7 or 8
Hit Hard 15 and Below Against a 2
Hit Hard 14 and Below Against a 3 and 4
Hit Hard 13 and Below Against a 5 and 6

Other changes in strategy are to hit all soft 18's against an ace, three card soft 18 against an 8, and hit three card 12 versus a 4.

Obviously there will be many other composition dependent exceptions to the conventional basic strategy which are not revealed by the infinite deck approximation to four deck or single deck play. So the reader feels he's getting his money's worth I will divulge the only four card hard 14 which should be hit against the dealer's five. You save .13% by hitting (T,2,A,A) v. 5 and now that you are armed with this information you can rush to Hong Kong and punish the casinos there by winning one extra bet out of every 35,000,000 hands you play!

In many of the casinos where "Five Card" appears, it collides with some of the other rule variations we have already discussed, creating a hydra-headed monster whose expectation cannot be analyzed in a strictly additive fashion. For instance, if we have already "early surrendered" 14 v dealer Ten, we can neither tie the dealer's natural 21 (allowed in Macao) nor turn it into a five card situation. The five card rule is a big money maker, though, being worth about .70% for four decks and a surprising .57% for one deck. (This is in Macao, where the player can "five card" his way out of some of the dealer's ten-up blackjacks.)

The following table gives the frequency of development of five card hands in a four deck game, with the one deck frequency in parentheses next to the four deck figure.

FIVE CARD HANDS DEVELOPED OUT OF 10,000 HANDS

Total \ Dealer Up Card	A,9,T	7,8	2,3,4,5,6
20	32 (31)		
19	31 (30)		13 (9)
18	34 (32)	28 (26)	12 (8)
17	33 (30)	29 (26)	12 (8)
16	22 (19)	19 (16)	8 (6)
15	14 (11)	11 (9)	5 (3)
14	8 (7)	6 (5)	3 (2)
13	4 (3)	3 (2)	2 (1)
12	2 (2)	1 (1)	1 (1)

Reflection justifies the closeness of the single deck frequencies to the four deck ones. A hand like (3,3,3,3,4), with repetition of a particular denomination, will be much less probable for a single deck, but (A,2,3,4,5), with no repetition, occurs more often in the single deck. Hands with only one repetition, like (2,3,4,4,5) are almost equally likely in either case and tend to make up the bulk of the distribution anyway.

When a bonus is paid for (6,7,8) of the same suit or (7,7, 7), different strategy changes are indicated depending on how much it is. We can use the infinite deck expectation table to approximate how big a bonus is necessary for (6,7,8) of the same suit in order to induce us to hit the 8 and 6 of hearts against the dealer's two showing. Suppose B is the bonus paid automatically if we get the 7 of hearts in our draw. We must compare our hitting expectation of

$$\frac{4}{52}(-.293-.293-.153+.122+.386+.640-6.000)+\frac{3}{52}(.882)+\frac{1}{52}B$$

with our standing expectation of $-.293$. The equation becomes $-.379+\frac{B}{52}=-.293$, with solution $B = 4.47$.

Hence, with a 5 to 1 bonus we'd hit, but if it were only 4 to 1 we'd stand.

Bonuses of fixed value, like a \$5 bonus for (A, J) of spades, are usually only of interest to the minimum bettor.

Double Exposure

Epstein* proposed a variation of blackjack called "Zweikartenspiel" in which the dealer's hole card is exposed but, as compensation, the house takes all ties. He gives a strategy for which a player expectation of 2.1% is quoted.

Apparently some casino personnel have read Epstein's book, for, in October of 1979, Vegas World introduced "Double Exposure", patterned after zweikartenspiel except that the dealer hits soft 17 and the blackjack bonus has been discontinued, although the player's blackjack is an automatic winner even against a dealer natural. The game is dealt from five decks and has an expectation of about −.3% for the player, who may only split his pairs once and should pursue the following strategy, depending on the dealer's hand.[E]

Against hard 4-10 play the page 18 strategy, except

a. don't double 8 v 5-6, 10 v 9, and 11 v 10

b. don't split (2,2), (3,3), and (7,7) v 4 and 7, (8,8) v 9-10, and (9,9) v 9

c. don't soft double 13-17 v 4 , 13-14 v 5, and 19 v 6

d. hit soft 18 v 8

e. stand on 16 v 7-10 and 15 v 10

Against 11 never double or split and stand on hard 14 and soft 19 or more

Against stiffs (12-16) don't risk busting, and

a. split all pairs but (5,5), except not (T,T) v 12

b. double soft 13-20, except not 20 v 12

c. double hard 5-11, except not 5-7 v 12-13

Against pat 17-20 play to win, never double, and only split (2,2), (3,3), (6,6), (7,7), and (8,8) v 17 and (9,9) v 18

Against soft 12-17

a. don't bust, except hit 12 v 12-13 and 12-17 v 17

b. double 11 v 12-17 and 10 v 14-16

c. hit soft 18, except not against 17

d. split aces, (8,8) v 16-17, and (9,9) v 15-16

*In private correspondence about the origin of the game, Epstein "graciously cedes all claim of paternity to Braun."

Atlantic City

"I said unto the fools,
Deal not foolishly"
Psalms 75:4

Until recently the early surrender rule was used in the six deck games played in Atlantic City, New Jersey. The dealer stands on soft 17, double after split is permitted, but pairs may be split only once.

An analysis of the player's expectation for these rules will be useful for illustrating how to employ the information in this chapter. To begin, we need an estimation of the six deck expectation for the typical rules generally presumed in this book. Interpolation by reciprocals suggests that the player's expectation will be one sixth of the way between .02% (single deck) and −.65% (infinite deck), and closer to the infinite deck figure. $-.65 + (.02 - -.65)/6 = -.65 + .11 = -.54\%$ is the presumed six deck expectation for the rules assumed in Chapter Two.

The right to double after split is worth .12%, while prohibition on resplits probably costs the player .04%, after rounding off. Early surrender itself provides a gain of .63%. Summarizing, we adjust the previous figure of −.54% by $.12 - .04 + .63 = .71(\%)$ and presume an advantage of .17% for basic strategy play. This truly philanthropic state of affairs led to much agony for the New Jersey casino interests!

Not only did the knowledgeable player have an advantage for a complete pack of 312 cards, but it turns out that the early surrender rule results in greater fluctuations in the player's advantage as the deck is depleted than those which occur in ordinary blackjack. An excess of aces and tens helps the player in the usual fashion when they are dealt to him, but the dealer's more frequent blackjacks are no longer so menacing in rich decks, since the player turns in many of his bad hands for the same constant half unit loss.

The effects of removing a single card of each denomination appear in the next table; even though Atlantic City games are all multiple deck the removals are from a single deck so comparisons can be made with other similar tables and methods presented in the book.

Effects of Removal on Early Surrender Basic Strategy Expectation (%)

A	2	3	4	5	6	7	8	9	T	Sum of Squares
−.68	.47	.55	.73	.92	.59	.34	−.06	−.28	−.64	4.53

These numbers are nearly proportional to those which describe the fluctuations in ordinary blackjack, so a system which was good for diagnosing advantage in Las Vegas would also have been good for this purpose in Atlantic City.

Unfortunately for the less flamboyant players who didn't get barred, a suit requiring casinos to allow card counters to play blackjack was ruled upon favorably by a New Jersey court. This had as its predictable result the elimination of the surrender option and consequently what had been a favorable game for the player became an unfavorable one. Under the new set of rules, in effect as of June 1981, the basic strategist's expectation is −.46%. (For the correct six deck basic strategy see the end of Chapter Eleven.)

The following chart of how much can be gained on each extra unit bet on favorable decks may be of some use to our East Coast brethren for whom "it's the only game in town."

Cards remaining	Gain for each unit bet in favorable situations (%)
286	.03
260	.07
234	.14
208	.19
182	.26
156	.34
130	.44
104	.56
78	.79
52	.99

Even with three fourths of the shoe dealt a three to one bet spread is necessary just to break even.

APPENDIX TO CHAPTER 8

A.

At one time I believed that the frequency of initial two card hands might be responsible for the difference between infinite and single deck expectations. However, multiplication of Epstein's single deck expectations by infinite deck probabilities of occurrence disabused me of the notion.

B.

One possible justification for the interpolation on the basis of the reciprocal of the number of decks can be obtained by looking at the difference between the infinite deck probability of drawing a second card and the finite deck probability. The probability of drawing a card of different denomination from one already possessed is $\frac{4k}{52k-1}$ for k decks and the corresponding chance of getting a card of the same denomination is $\frac{4k-1}{52k-1}$ . . The differences between these figures and the constant $\frac{1}{13}$, which applies to an infinite deck, are $\frac{1}{13(52k-1)}$ and $\frac{12}{13(52k-1)}$ respectively. These differences themselves are very nearly proportional to the reciprocal of the number of decks used.

C.

The table comparing fluctuations in various numbers of decks was created by equating the expressions $\sqrt{\frac{(N-n)}{(N-1)n}}$ for $N = 52, 104, 208, 312,$ and ∞. In the last case we find ourselves in the position of

. . . a mathematician from Trinity
Who took square roots of infinity.
But because of the strain
That it put on his brain,
He chucked math and took up Divinity.

D.

Different equations are necessary to evaluate early surrender for different hands from one and four decks. For instance, with (T,2) v. A, our equations are $\frac{34}{49}E - \frac{15}{49} = -\frac{1}{2}$ for one deck and $\frac{142}{205}E - \frac{63}{205} = -\frac{1}{2}$ for four decks.

The solutions, −.2794 and −.2782, differ very little from the infinite deck value of −.2778.

E.

The player's loss of ties is greatly offset by aggressive splitting and doubling to exploit the dealer's visible stiff hands. The elimination of the blackjack bonus renders the ace inconsequential for bet variation, as the following table of effects of removal on basic strategy advantage (in %) shows.

A	2	3	4	5	6	7	8	9	T
.11	.43	.60	.85	1.11	.59	.32	−.15	−.51	−.78

The removals are scaled as if they were from a single deck so comparisons and calculations can be made as in Chapter Six. The magnitudes show Double Exposure to be far more volatile than ordinary blackjack.

There are surprisingly many two card, composition dependent, exceptions to the page 126 strategy: stand with (A,7) v (8,3) and (7,6) and (8,5) v hard 11, except hit (8,5) v (9,2); double 7 v hard 13 other than (T,3); hit (T,6) v (6,2) and (9,7) v hard 7.

9
MISCELLANY

". . .the gallant all-or-nothing of the gambler, who hates the little when he can not have the much, and would rather stalk from the tables clean-picked than suffer ruin to be tickled by driblets of the glorious fortune he has played for and lost."
George Meredith, **The Egoist**

A variety of reasonable criteria dictate that the player's optimal bet should be proportional to his advantage on the hand about to be dealt. Practical casino conditions, however, make this impossible. For one thing, a negative wager (equivalent to betting on the house when they have the edge) is not permitted. Also, the discreteness of money and allowable wagers does not coincide with the mathematical ideal of infinitely divisible capital—try betting $2.74 at Caesars Palace![A,B]

But the major barrier to such perfectly scaled wagering is that it quickly tips off the casino to the player's identity as a card counter. When I first started playing, I religiously ranged my bets according to Epstein's criterion of survival. Going from $1 to $5 to $11 to $2 was not an uncommon pattern for me until I came to realize I was paying a far higher price in casino countermeasures than the theoretical minimization of my ruin probability was worth. (Besides, when the truly degenerate gambler is wiped out of one bank he need only go back to honest work for a few months until he has another.)

In my opinion the entire topic has probably been overworked. The major reason that such heavy stress has been placed on the problem of optimal betting is that it is one of the few which are easily amenable to solution by existing mathematics, rather than because of its practical importance.

Nevertheless, we can gain some insight into the situation by contriving a simplified, variable advantage, compound game which approximates blackjack. Suppose Greta Gross and Opie Optimal both are required to bet at least one unit on each play of a game which has a 2% disadvantage 60% of the time, a 2% advantage 30% of the time, and a 6% advantage 10% of the time. The game resembles basic strategy blackjack with about 28 cards left in the deck, since for flat bets it is an even game, but every extra unit bet in favorable situations will earn 1.2% of a unit per hand.

Now, both Greta and Opie know before each play which situation they will be confronting. Opie bets optimally, in proportion to her advantage, 2 units with a 2% advantage and 6 units with the 6% edge, while Greta bets grossly, 4 units whenever the game is favorable. Thereby they both achieve the same 3.6% of a unit expectation per play. Starting with various bank sizes, their goals are to double their stakes without being ruined. The results of 2000 simulated trials in each circumstance appear below.

NUMBER OF TIMES RUINED TRYING TO DOUBLE A BANK OF

	20	50	100	200
Opie	877	668	438	135
Greta	896	733	541	231

Greta is obviously the more often ruined woman, but since they have the same expectation per play there must be a compensating factor. This is, of course, time—whether double or nothing, Greta usually gets her result more quickly. This illustrates the general truth (pointed out by Thorp in his Favorable Games paper) that optimal betting systems tend to be "timid", perhaps more so than a person who values her time would find acceptable.[C]

Again, the necessity of camouflage in real casino play seems to make academic any consideration of precisely scaled wagers—you just shove out as much as you feel you can get away with when the deck gets rich. If you want a system designed to maximize the intensity of your involvement, you might try betting 2/17 or 2/9 of your capital on each favorable deck (depending, of course, on whether you can or cannot double after splitting). You play every hand as if it's your last, and it might be, if you lose an insurance bet and split four eights in a losing cause![D]

More than one Hand?

Another common concern voiced by many players is whether to take more than one hand. Again, practical considerations override mathematical theory since there may be no empty spots available near you.

(A bit of rather amusing advice on this matter appeared in a book sold commercially a few years ago. The author stated that "by taking two hands in a rich situation you reduce the dealer's probability of getting a natural." This more-the-merrier approach was contradicted in another section where advice was given to avoid playing with other people at the table because they "draw off the cards which would have busted the dealer."

The curious view that "probability runs in streaks" was also espoused. This brings to mind how so many, even well regarded, pundits of subjects such as gambling, sports, economics, etc. confuse their own verbal reaction to a past event with an actual explanation of it and an augur for the future. Thus, we have the gambling guru who enjoins us to "bet big when you're winning," the sports announcer who feels compelled to attribute one team's scoring of several consecutive baskets to the mysterious phantom "momentum," and the stock market analyst who cannot report a fall in price without conjuring up "selling pressure."

Other gems of wisdom in this $25 volume included the preposterous claim that "the player has a 1% advantage off the top for the complete deck." At the end the reader is offered special lessons (at $100 per hour and by approval only, to be sure) to learn "super-attenuated" play. A trip to the dictionary confirms that this latter description is probably the most accurate in the book. But to debunk mountebanks is to digress.)

It's best not to confuse the issue by discussing certain obvious strategy advantages (particularly from insurance) which might accrue from the taking of several hands. Nevertheless, there can be a certain reduction in fluctuations achievable by playing multiple spots.

Suppose we have our choice of playing from one to seven hands at a time, but with the restriction that we have the same amount of action every round (every dealer hand). For instance, we might contemplate one hand for $420, two at $210 each, . . . or seven at $60 apiece. Then the following table shows the *relative* fluctuation we could expect in our capital if we follow this pattern over the long haul.

Number of Hands	1	2	3	4	5	6	7
Relative Fluctuation	1.44	1.20	1.11	1.06	1.03	1.01	1.00

However, there's another equally plausible perspective which reverses this. Assuming that we play each of our hands as fast as the dealer does his and ignoring shuffle time, then we can play a single spot on four rounds as often as seven spots on one round. Similarly three spots could be played twice in the same amount of time. Now, with our revised criterion of equal total action per time on the clock, our table reads:

Number of Hands	1	2	3	4	5	6	7
Relative Fluctuation	.72	.74	.79	.84	.90	.95	1.00

As we see with so many blackjack questions the answer depends on what qualifications are attached. Of course, all this ignores the fact that taking more hands requires more cards and might trigger shuffle up on the dealer's part if he didn't think there were enough cards to complete the round. Or, sometimes there would be enough cards to deal once to two spots but not twice to one spot. One rarely knows in advance precisely what the dealer will do.[E,F]

Few casinos will deal more than two rounds if five spots are being played, but when it happens it can be very unfavorable, ironically despite the fact we're getting deeper into the deck. It's been my observation that when this third round is dealt to five players it's almost always because the first two rounds used very few (and predominantly high) cards; hence the remainder of the deck is likely to be composed primarily of low ones. This practical example, which I've witnessed more than I should have, is related to the effect of a fixed shuffle point which is of interest to blackjack simulators and discussed in the chapter appendix.[G]

Shuffling

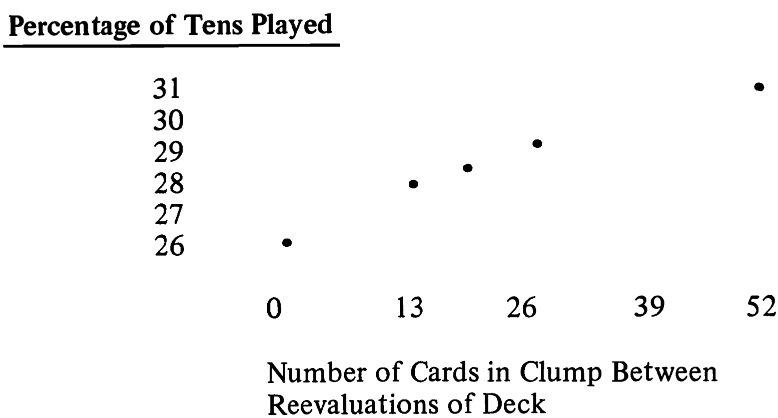
A related consideration is the practice of "preferential shuffling" wherein the dealer shuffles the deck whenever he perceives it (for whatever reason) as favorable. A good example to illustrate the truncated distribution which results can be obtained by reverting to a simplistic, non-blackjack example.

Consider a deck of four cards, two red and two black. As in Chapter Four, the dealer turns a card; the player wins if it's red and loses on black. Ostensibly we have a fair game, but now imagine an oblivious, unsuspecting player and a card-counting, preferentially shuffling dealer. Initially there are six equally likely orderings of the deck.

R	R	B	B
R	B	R	B
R	B	B	R
B	B	R	R
B	R	B	R
B	R	R	B

Since the dealer is trying to keep winning cards from the player, only the enclosed ones will be dealt. The effect, we see, is the same as playing one hand from a deck of 14 cards, 9 of which are black. Instead of having an even game the player is at a disadvantage of 29%. As an exercise of the same type the reader might start with a five card deck, three red and two black. Could preferential shuffling overcome the apparent 20% basic strategy player advantage?[H]

Bringing the discussion back to blackjack, we might wonder how well the dealer who counted tens could do at keeping them away from the player by shuffling away all ten-rich decks. The answer depends on how often the deck is reevaluated; blackjack uses typically four to twenty-four cards per round, depending on the number of players. The following chart shows the percentage of tens that would be dealt as a function of the size of the clump of cards the dealer observes before making his next decision on whether to reshuffle.



Since about five or six cards are usually used against a single player we can conclude that the dealer could reduce the proportion of tens dealt to about 26.5% in head on play. This would give the basic strategist a 1.5% disadvantage. By using a better correlated betting count to decide when to reshuffle, the house edge could probably be raised to 2%.

Since mathematically this is equivalent to shorting the deck (in the previous example to 13 Tens/49 cards), and the latter practice is specifically prohibited by law, some people have suggested that the Gaming Commission should regard the practice as illegal. In all honesty, though, I think we must recognize that player card-counting is just the obverse of preferential shuffling—what's sauce for the goose is also for the gander.

While on the subject, it might be surprising that, occasionally, the number of times the dealer shuffles may influence the player's expectation. New decks all seem to be brought to the table with the same arrangement when spread:

A23...QKA23...QKKQ...2AKQ...32A.

If the dealer performs a perfect shuffle of half the deck against the other half, then, of course, the resultant order is deterministic rather than random. Three perfect shuffles of a brand new deck give the head-on basic strategist about a 30% advantage (where the cards are cut is considered as uniformly random), whereas five perfect shuffles reverse the edge to 25% in favor of the house! Is it a coincidence that one of the major northern Nevada casinos has a strict procedure calling for five shuffles of a new deck, but three thereafter?

Even experienced dealers would have some difficulty trying to perform five perfect shuffles (a "magician" demonstrated the skill at the Second Annual Gambling Conference sponsored by the University of Nevada), but to get some idea of what might happen if this were attempted, I asked a professional dealer from the Riverside in Reno to try it. He sent me the resultant orderings for eight such attempts. These had the basic strategy player losing 29 units in 336 hands, a 9% disadvantage. (42 hands were dealt from each deck, assuming a cut must have at least five cards in the smaller part.) Although a result of this sort is not particularly significant in that it, or something worse, would occur about 7% of the time by chance alone, none of the eight decks favored the player.

Previous Result's Effect on next Hand

Blackjack's uniqueness is the dependence of results before reshuffling takes place. While the idea that a previous win or loss will influence the next outcome is manifest nonsense for independent trials gambles like roulette, dice, or keno, it is yet conceivable that in blackjack some way might be found to profitably link the next bet to the result of the previous one.

Wilson discusses the intuition that if the player wins a hand, this is evidence that he has mildly depleted the deck somewhat of the card combinations which are associated with him winning, and hence he should expect a poorer than average result next time. My resolution to the question, when it was first broached to me, was to perform a Bayesian analysis through the medium of the Dubner Hi Lo index. This led to the tentative conclusion that the player's expectation would be reduced by perhaps .10% on a hand dealt following a win and before a reshuffle.[I]

This .10% figure has been experimentally confirmed by John Gwynn's humongous computer simulation of basic strategy play. Gwynn also found that a push on the previous hand is apparently a somewhat worse omen for the next one than a win is.

It follows, then, that the player's prospects must improve following a loss, although of course not much, certainly not enough to produce a worthwhile betting strategy. When all is said and done, the most immediate determiner of the player's advantage is the actual deck composition he'll be facing, and knowledge of whether he won, lost, or pushed the last hand, in itself, really tells us very little about what cards were likely to have left the deck, and implicitly, which ones remain.

APPENDIX TO CHAPTER 9

A.

Epstein proposes minimizing the probability of ruin subject to achieving an overall positive expectation. This "criterion of survival" dictates a wager of $k \cdot \log \left(\frac{p}{(1-p)}\right)$ for values of p, the single trial probability of success, which are greater than a critical $p^* > \frac{1}{2}$, p^* being determined by an integral equation. The constant k equals $-1/\log\ ((1-p^*)/p^*)$.

If we let $a=2p-1$ be the player's advantage in the subgame characterized by p, we have

$$\log\left(\frac{p}{(1-p)}\right) = \log\left(\frac{\frac{1}{2}+\frac{a}{2}}{\frac{1}{2}-\frac{a}{2}}\right) = \log\left(\frac{1+a}{1-a}\right)$$

Since p will generally be close to 1/2, a will be small and $\log\left(\frac{1+a}{1-a}\right) \sim \log(1+2a) \sim 2a.$ The first approximation comes from discarding higher order terms in

$\frac{1+a}{1-a} = (1+a)(1+a+a^2+\cdots)$ and the second one from the expansion $\log(1+x) = x - x^2/2 + x^3/3 \cdots.$

The important conclusion is that the optimum wager for survival is approximately proportional to the player advantage. Thus it is generally consistent with the famous Kelly criterion for maximizing the exponential rate of growth.

B.

Another reasonable principle which leads to proportional wagering is that of minimizing the variance of our outcome subject to achieving a fixed expectation per play. Suppose our game consists of a random collection of subgames (indexed by i) occurring with probability P_i and having corresponding expectation E_i. Imagine that we are required to bet at least one unit always (even when $E_i<0$), but otherwise are free to vary our wagers, $W_i \geqslant 1$.

If we bet one unit when $E_i \leqslant 0$, W_i otherwise when $E_i>0$, our expectation per play will be

$$E = \sum_{E_i \leqslant 0} P_i E_i + \sum_{E_i>0} W_i P_i E_i$$ and our variance will be

$$V = \sum P_i W_i^2 - E^2 = \sum_{E_i \leqslant 0} P_i + \sum_{E_i>0} P_i W_i^2 - E^2 .$$

Thus to choose the optimal W_i we have a problem for LaGrange's multipliers.

With $E_i>0$ understood for all summations, we must minimize $\sum P_i W_i^2$ subject to $\sum W_i P_i E_i = C$. We form $L = \sum P_i W_i^2 + \lambda\left(\sum P_i E_i W_i - C\right)$;

then $\dfrac{\partial L}{\partial \lambda} = \sum P_i E_i W_i - C = 0 \Longrightarrow C = \sum P_i E_i W_i$ and

$$\frac{\partial L}{\partial W_i} = 2P_i W_i + \lambda P_i E_i = 0 \Longrightarrow W_i = -\frac{\lambda}{2} E_i .$$

Again we see that the optimal wager is proportional to the expectation of the subgame.

C.

The actual difference equation which governs, for instance, Greta's probability of being ruined before doubling her stake is

$$P_x = .306P_{x-1} + .294P_{x+1} + .206P_{x+4} + .194P_{x-4}$$

Greta's averaged squared wager is 7, and we may attempt to approximate her experience by having her bet $\sqrt{7}$ units on the toss of a biased coin with P(heads)=.5 + .036/2 $\sqrt{7}$ and thereby achieve the same expectation and variance.

Opie's difference equation is of order 12, and even more intractable. She would have the same mean and variance if she played a coin tossing game with P(heads)= .5 + .036/2 $\sqrt{5.4}$ and a wager of $\sqrt{5.4}$ units. Increasing their bets effectively diminishes their capital, and when this is taken into account we come up with the following approximations to the ideal frequencies of their being ruined, in startling agreement with the simulations.

	20	50	100	200
Opie	867	678	417	130
Greta	897	748	527	226

Similarly, if EX is the expectation and EX^2 is the averaged squared result of a blackjack hand, then betting $\sqrt{EX^2}$ units on a coin toss with p = .5 + EX/2 $\sqrt{EX^2}$ will give us a game with the same mean and variance of a single unit bet on the blackjack hand. With this formulation we can approximate gambler's ruin probabilities and also estimate betting fractions to optimize average logarithmic growth, as decreed by the Kelly criterion.

Remembering, again, that betting $\sqrt{EX^2}$ units changes our capital proportionately, we estimate that betting EX/EX^2 of the player's current bankroll maximizes the expected logarithm of growth among all fixed fraction betting strategies. It is interesting that this intuitive approach also appears as a consequence of using a two term Taylor series for $\log(1+B_j f)$, where $B_j=0, \pm 1, \pm 2, 1.5, \pm 3, \pm 4$ are the possible blackjack payoffs and f is the fraction of capital bet. In

Chapter Eleven the value 1.26 is suggested for EX^2 with a full deck. It seems doubtful that it would vary appreciably as the deck composition changes within reasonable limits.

Certain properties of long term growth are generally appealed to in order to argue the optimality of Kelly's fixed fraction betting scheme, and are based on the assumption of one bankroll, which only grows or shrinks as the result of gambling activity. The questionable realism in the latter assumption, the upper and lower house limits on wagers, casino scrutiny, and finiteness of human life span all contribute to my lack of enthusiasm for this sort of analysis.

D.

The example of betting 2/17 of your capital illustrates the inadequacy of conventional gambler's ruin formulas, which are based on a single unit either won or lost at each play, for blackjack. Precisely the suggested scenario could unfold: a hand could be dealt from a residue of one seven; one nine; four threes, eights, and aces each; and ten tens. This would have a putative advantage of about 12%, and call for a bet in this proportion to the player's current capital.

In fact, ignoring the table limits of casinos, the conjectured catastrophe would be *guaranteed* to happen and ruin the player sooner or later. This is opposed to the Kelly idealization wherein, with only a fixed proportion of capital risked, ruin is theoretically impossible.

E.

From simulated hands I estimate the covariance of two blackjack hands played at the same table to be .50. Since the variance of a blackjack hand is about 1.26 squared units, we have the formula $V(n) = 1.26n + .50n(n-1)$ for the variance of the result when one unit is bet at each of n spots.

F.

The first table of relative fluctuation is obtained by multiplying $V(n)$ by $\left(\frac{7}{n}\right)^2$ and then taking the square root of the ratio of this quantity to $V(7)$. The second table arises from taking $\sqrt{\frac{V(n)(n+1)49}{8n^2\ V(7)}}$ and incorporates the $8/(n+1)$ rounds of play in the time it takes one player to play seven spots and the $\left(\frac{7}{n} \cdot \frac{n+1}{8}\right)$ adjustment to guarantee they have the same total action.

G.

One of the problems encountered in approximating blackjack betting situations is that the normal distribution theory assumes that all subsets are equally likely to be encountered at any level of the deck. An obvious counterexample to this is the 48 card level, which will occur in real blackjack only if the first hand uses exactly four cards.

More than half of these cases would result from either the player or the dealer having blackjack and Gwynn's simulations showed that the player suffered a little more than a 1% depression in advantage when a hand was dealt with 48 cards. The only imaginable favorable situation which could occur then would be if the player stood with something like (7, 5) v 6 up, A underneath.

The other problem is the "fixed shuffle point" predicted very well by David Heath in remarks made during the Second Annual Gambling Conference at Harrah's, Tahoe. Gwynn's simulations, using a rule to shuffle up if 14 or fewer cards remained, confirmed Heath's conjecture quite accurately. Roughly speaking, almost every deck allowed the completion of seven rounds of play, but half the time an eighth hand would be played and it tended to come from a deck poor in high cards, resulting in about a 2.5% depression in player advantage for this occasional "extra" hand.

Thus there was a difficulty in my approximation to the Gwynn results. I resolved it by assuming a bet diagnosis arose every 5.4 cards for the first seven hands, and that every other time, an eighth hand, drawn from a pack with a 2.5% reduction in basic advantage, occurred with 16 cards remaining.

H.

Preferential shuffling presents an interesting mathematical problem. For example, if the preferential shuffler is trying to keep exactly one card away from the player, he can deal one card and reshuffle if it isn't the forbidden card, but deal the whole deck through if the first one is. The player's chance of getting the particular card is thus reduced from 1/52 to 1/103 and this halving of true probability seems to be the most extreme distortion possible.

I.

I carried out the Bayesian analysis by using an a priori Hi Lo distribution of points with six cards played and a complicated formula to infer hand-winning probabilities for the different values of Hi Lo points among the six cards assumed used. From this was generated an a posteriori distribution of the Hi Lo count, assuming the player did win the hand. A player win was associated with an average drain of .18 Hi Lo points for the six cards used, and hence the remaining deck would tend to have a −.18 count with 46 cards remaining. This translates into about a .10% depression of player advantage.

10

CURIOS AND PATHOLOGIES IN THE GAME OF TWENTY-ONE

There are strange things done in the Vegas sun
By the men who moil for gold;
The Nevada trails have their secret tales
That would make your blood run cold;
The casino lights have seen queer sights;
But the queerest they ever did see
Was the night of the show at Lake Tahoe
That I split two fives against a three.
—Profuse apologies to Robert Service

Card counters are like the prototypical Don Juan who wants every woman he meets to succumb to him and then wishes to marry a virgin; they want all fives to be out of the deck before they raise their bets and then they want the dealer to show one as up card! There is an apparent paradox in that the cards whose removal most favors the player before the deal are also the cards whose appearance as dealer's up card most favors the player.

The following table compares (in %) the effect of removal on basic strategy favorability with the player's advantage when the corresponding card is shown by the dealer. Thus an intuitive understanding of the magnitude and direction of the effects is not easy to come by. The last line tabulates the player's expectation as a function of his own initial card and suggests a partial explanation of the "contradiction", although the question of why the player's first card should be more important than the dealer's is left open.

	A	2	3	4	5	6	7	8	9	Ten
Effect of Removal	-.61	.38	.44	.55	.69	.46	.28	0	-.18	-.51
Player's Advantage when Dealer Exposes	-36	10	14	18	24	24	14	5	- 4	-17
Player's Advantage When his First Card is	52	-12	-14	-16	-19	-18	-17	-9	0	13

An intriguing resolution can be provided by the definition of three simple variables and a single linear equation which enables us to account for the basic strategy favorability not in terms of what cards *are in the deck*, but rather, in terms of what the cards *in the deck do*. Let X_1 be the number of stiff totals (12-16) which will be made good by the particular denomination considered and X_2 be the number of stiff totals the card will bust. Finally, to mirror a card's importance in making up a blackjack, define an artificial variable X_3 to be equal to one for a Ten, four for an Ace, and zero otherwise. The following equation enables prediction of the ultimate strategy effects with a multiple correlation of .996.

$$Y = .14 \cdot X_1 - .07 \cdot X_2 - .18 \cdot X_3$$

CARD	X_1	X_2	X_3	PREDICTED	ACTUAL EFFECT
A	1	0	4	-.58	-.61
2	2	0	0	.28	.38
3	3	0	0	.42	.44
4	4	0	0	.56	.55
5	5	0	0	.70	.69
6	4	1	0	.49	.46
7	3	2	0	.28	.28
8	2	3	0	.07	.00
9	1	4	0	-.14	-.18
Ten	0	5	1	-.53	-.51

Some Extremely Interesting Facts

Like the man who wants a dog who's shaggy, but not too shaggy, the ten counter is also difficult to satisfy. He wants the deck to be rich in tens, but not too rich. Some authors, who try to explain why an abundance of tens favors the player, state that the dealer will bust more stiffs with ten rich decks. This is true, but only up to a point. The dealer's probability of busting, as a function of ten density, appears to maximize (.295) with about 41% tens in the deck. This compares with a normal .286 and a .185 for a deck with no tens at all.

The player's advantage, as a function of increasing ten density, behaves in a similar fashion, rising initially, but necessarily returning to zero when there are only tens in the deck and player and dealer automatically push with twenty each. It reaches its zenith (almost 13%) when 73% of the cards are tens. Strangely, a deck with no tens also favors the player who can adjust his strategy with sufficient advantage to overcome the $(.185)^2 = .034$ double bust factor, which quantity is the basis for whatever advantage the casino may enjoy.

Thorp presents the classic example of a sure win with (7,7,8,8,8) remaining for play, one person opposing the dealer. (7,7,7,7,8,8,8,8) gives a higher expectation of 120% but allows the possibility of a loss. This may be the richest (highest expectation) subset of a 52 card deck. An infinite deck composition of half aces and half tens maximizes the player's chance for blackjack but gives an expectation of only 68% whereas half sevens and half eights will yield an advantage of 164%. These figures are arrived at using the assumption that, except for aces, up to four cards may be split; with this proviso a deck of all twos, all threes, or all eights would provide a profit of 400% per hand.

An ordinary pinochle deck would give the player about a 45% advantage with proper strategy, assuming up to four cards could be split. Insurance would always be taken when offered; hard 18 and 19 would be hit against dealer's ten; and, finally, (A,9) would be doubled and (T,T) split regardless of the dealer's up card.

The Worst Deck

Conversely one might wonder what subset of cards, regardless of size, produces the greatest player disadvantage when optimal strategy is conducted against it. (Certainly a deck of all fives would be devastating to the basic strategist who would be forever doubling down and losing, but optimal play would be to draw to twenty and push every hand.) For a "large" deck (where player's and dealer's hands may be regarded as independent) the player disadvantage can never be less than $2\sqrt{2}-3=-17.2\%$. This conclusion results from imagining a deck where the dealer's chance of busting is $\sqrt{2}-1=.414$. If the dealer busts less frequently than this, "mimic the dealer" strategy gives an expectation better than $-(\sqrt{2}-1)^2=2\sqrt{2}-3$, while if the dealer busts more often the player can adopt a "never bust" strategy and expect at least $(\sqrt{2}-1)-(1-(\sqrt{2}-1))=2\sqrt{2}-3$.

The results of a program written to converge to the worst possible composition of an infinite deck suggest this lower bound can never be achieved. A -12.5% disadvantage was reached with the bizarre composition of 61.17% twos, 20.35% sixes, and 18.48% tens. No odd totals are possible and the only "good" hands are 18 and 20. The player cannot be dealt hard ten and must "mimic the dealer" with only a few insignificant departures (principally standing with 16 against Ten and splitting sixes against dealer two and six). The dealer busts with a probability of .357 and the $(.357)^2=.1277$ double bust probability is very nearly the house advantage. The creation of this pit boss's delight (a dealing shoe gaffed in these proportions would provide virtual immunity from the depredations of card counters even if they knew the composition) may be thought of as the problem of increasing the dealer's bust probability while simultaneously leaching as many of the player's options from "mimic the dealer" strategy as possible.[A]

Effect of Removal on Dealer's Bust Probability

Surprise is often expressed at the anomalous fact that removal of a seven makes hitting 16 vs. Ten less favorable. It can be verified that proper strategy with a sevenless deck is to stand in this situation and a thought experiment should convince the reader that as we add more and more sevens to the deck we will never reach a point where standing would be correct: suppose four million sevens are mixed into an otherwise normal deck. Then hitting 16 will win approximately four times and tie once out of a million attempts, while standing wins only twice (when dealer has a 5 or 6 underneath) and never ties! (Calculations assume the occurrence of two nonsevens is a negligible second order possibility.) The addition of a seven decreases the dealer's chance of busting to more than offset the player's gloomier hitting prognosis.

In the following table we may read off the effect of removing a card of each denomination on the dealer's chance of busting for each up-card. The last line confirms that the removal of a seven increases the chance of busting a ten by .60%, which is a more extreme change than that produced by any other card.

EFFECT OF REMOVAL ON DEALER'S CHANCE OF BUSTING (in %)

Dealer's Up Card	DENOMINATION REMOVED A	2	3	4	5	6	7	8	9	T	Chance of Bust	Sum of Squares
A	- .08	-.01	.04	.14	.24	1.00	.70	.41	.11	-.64	17.05	3.4
2	.47	.07	.08	.17	.90	.75	.60	.44	.23	-.93	35.23	5.7
3	.44	.05	.14	.85	.99	.84	.68	.48	-.72	-.94	37.44	7.4
4	.39	.10	.81	.95	1.07	.93	.70	-.49	-.70	-.94	39.33	8.5
5	.34	.79	.91	1.04	1.15	.93	-.27	-.51	-.71	-.92	41.74	9.1
6	1.31	.81	.93	1.03	1.13	- .06	-.29	-.47	-.66	-.93	42.14	9.8
7	.67	.23	.34	.45	- .01	- .24	-.45	-.61	-.87	.13	26.44	2.3
8	.59	.22	.32	- .11	- .11	- .32	-.47	-.74	.26	.09	24.60	1.5
9	.52	.22	-.22	- .21	- .18	- .37	-.60	.40	.22	.06	23.12	1.2
T	.11	-.31	-.28	- .27	- .24	- .48	.60	.41	.23	.06	23.20	1.1

We can use the methods proposed in Chapter Six to estimate the dealer's chance of busting for various subsets. What we learn from the magnitudes of numbers in the "Sum of Squares" column is that the probability of busting tens and nines fluctuates least as the deck is depleted, while the chance of breaking a six or five will vary the most. This is in keeping with the remarks in Chapter Three about the volatility experienced in hitting and standing with stiffs against large and small cards.

The World's Worst Blackjack Player

Ask "who is the best blackjack player?" and you can bet there will be a great gnashing of egos among the various entrepreneurs and publicity seekers attracted to the game. Watching a hopeless swain stand with (3,2) v T at the Barbary Coast in Las Vegas rekindled my interest in the question "who is the world's worst player and how bad is he?"

Since few are masochistic enough to deliberately bust all hands or double down on all totals over eleven, we can put a more realistic upper bound on the world's worst player by selecting from the following smorgasboard of prices paid for departure from basic strategy.

Play	Penalty (in %)
Always insure blackjack	.01
Always insure (T,T)	.05
Always insure anything	.23
Stand on stiffs against high cards	3.
Hit stiffs against small cards	3.2
Never double down	1.6
Double ten v T or A	.05
Always split and resplit (T,T)	8.
Always split (4,4) and (5,5)	.4
Other incorrect pair splits	.2
Failure to hit soft 17	.3
Failure to hit soft 18 v 9 or T	.03
Failure to hit (A,small)	2.

Hence it seems unlikely that any but the deliberately destructive could give the house more than a 15% edge. This is only a little more than half the keno vigorish of 26%: the dumbest blackjack player is twice as smart as any keno player!

Observations I made in the spring of 1987 showed that the overall casino advantage against a typical customer is about 2%. The number and cost of players' deviations from basic strategy were recorded for 11,000 hands actually played in Nevada and New Jersey casinos. The players misplayed about one hand in every 6.5, at an average cost of 9% per mistake. This translates into an expectation 1.4% worse than basic strategy which, for typical multiple deck games, gives a 2% casino edge. Other findings: Atlantic City players were closer to basic strategy than those in Nevada, by almost .5%. The casinos probably win less than 1.5% of the *money* bet, this because a better quality of play is associated with the large wagers of high rollers.

Incidentally, standing with (A,4) v T is more costly by 13% than standing with (3,2). It's only because we've grown more accustomed to seeing the former that we regard the latter as the more depraved act. One player, when innocently asked why he stood on (A,5), replied "Even if I *do* get a ten (emphasis to indicate that he apparently thought this was the best of all possible draws) I still would only have 16".

The Unfinished Hand

Finally, let the reader be apprised of the possibility of an "unfinished" blackjack hand. Many casinos permit the splitting of any ten-valued cards, and of these, some allow unlimited splitting (this assumption isn't vital in what follows if there are four players at the table). Imagine a player who splits sixteen tens and achieves a total of twenty-one on each hand by drawing precisely two more cards. The dealer necessarily has an ace up, ace underneath, but cannot complete the hand. By house rules she is condemned throughout eternity to a Dante's Inferno task of shuffling the last two aces, offering them to the player for cut, attempting to hit her own hand, and rediscovering that they are the burn and bottom cards, unavailable for play!

APPENDIX TO CHAPTER 10

A.

Minimization of a function of ten variables is not an easy thing to do. (In this case the ten variables are the densities of the ten distinct denominations of cards and the function is the associated player advantage.) Although I cannot prove this is the worst deck, there are some strong arguments for believing it is:

1. The minimum of a function of many variables is often found on the boundary and with seven denominations having zero densities we definitely are on a boundary.
2. To approach the required $\sqrt{2}-1=41\%$ dealer bust probability for the theoretically worst deck there would have to be some eights, nines, or tens. If there are eights or nines, their splitting would probably provide a favorable option to "mimic the dealer" strategy which would reduce the 17% disadvantage from standing with all hands. Also, if there are nines or tens, the player will occasionally, with no risk of busting, reach good totals in the 17 to 21 range, thus achieving a better expectation than "never bust" strategy was assumed to yield. Either way, the theoretical -17% is almost certainly not achievable.
3. There's an intuitive argument for having only even cards in the "worst deck" — once any odd card is introduced then all totals from 17 to 26 can be reached. Half of these are good and half bad. But with only even cards you can only reach 18, 20, 22, 24, and 26, three out of five of which are busts. This reduced flexibility should help in raising the dealer bust probability while simultaneously minimizing the player's options.
4. Assuming only even numbers, the eights are filtered out because they provide favorable splits for the player.

5. The fours make good any totals of 14 and 16 and hence lower the dealer bust probability.
6. The twos are tantalizers in that they bring home only totals of 16 for the dealer, but keep other stiffs stiff for another chance of being busted.

At one of his seminars, the author instructs Sue of the Sacramento Zoo in the art of playing natural 21.

11

SOME TECHNIQUES FOR BLACKJACK COMPUTATIONS

—To iterate is human,
To recurse is divine.—
Source Obscure

Previous blackjack investigations have incorporated simulations or other approximations to evaluate the player's basic strategy from pair splitting. This has been due largely to a concern for playing the subsequently derived hands optimally, depending on the cards used on earlier parts of the split. This begs a distinction between "basic" and "zero-memory" strategy, and will lead us to an algorithm for exact determination of repeated pair splitting expectation with zero-memory strategy.

Imagine you are playing single deck blackjack and have split three deuces against a four. To each of the first two deuces you draw two sevens, and on the third deuce you receive a ten. It is basic play to hit (T,2) vs. 4; however, if you take cognizance of the four sevens and two deuces on the table but not in the hand you contemplate, you recognize a 6% gain by *not* hitting. Should this 6% gain be assigned to *zero-memory* pair splitting expectation?

If you answered yes to the previous question, suppose the first two deuces were busted with two tens each. You are dealt an ace and a nine to the third deuce. It is basic play to stand with (A,2,9) vs 4, but if you *remember* the four tens and two deuces the dealer picked up when you busted then you recognize a 7% advantage in hitting. Now answer the previous question.

One may take the view that zero-memory expectation excludes such possible gains from varying strategy on subsequently derived hands on the grounds that the storage of zero-memory takes into account only the cards composing the hand to be acted on, rather than other hands (the result of previous pair splits) already resolved. Indeed Epstein suggests that zero-memory implies knowledge only of the player's original pair and dealer's up card.

Now, consider splitting eights against a seven in a single deck:

A. Calculate the conditional expectation for starting a hand with an eight against dealer's seven given that 1. two eights have been removed from the deck and 2. player cannot draw any eight as *first* card to his eights.

B. Calculate etc. given 1. three eights removed and 2. as above.

C. Calculate etc. given 1. all eights removed and 2. as above.

The second condition in A. and B. will require an intricate readjustment of probabilities for drawing to both the player's and dealer's hands, since the first card on any of the other split eights is known to be a non-eight. For example, if exactly two eights were split and (8,2,9) was developed on the hand being played out, the dealer's chance of having an eight underneath would be 2/46 rather than the 2/47 we might presume. Similarly the dealer's chance of having a five underneath would be 44/46 x 4/45 rather than 4/47.

The player's expectation from repeated pair splitting is now given by

$$\frac{1081\cdot(2\cdot A)}{1176} + \frac{90\cdot(3\cdot B)}{1176} + \frac{5\cdot(4\cdot C)}{1176}$$

The three fractions are, of course, the probabilities of splitting two, three, and four eights.

The extension to two and four decks is immediate. Let E(I) be the previously described conditional expectation if exactly I cards are split (and hence removed) and P(I) be the probability that I cards will be split; then the pair splitting expectation is:

$$\sum_{I \geqslant 2} P(I) \cdot I \cdot E(I)$$

The coefficients, I · P(I), shrink rapidly and a very satisfactory estimate of E(J) for $J \geqslant 3$ could be achieved by extrapolation from the calculated value of E(2). To do this we introduce an artificial E(1) (without any reference to pair splitting), as the weighted average expectation of the hands (8,A) (8,2). . . (8,7), (8,9), (8,T). These expectations would already be available from the general blackjack program and provide us with the base point for our extrapolation. For infinite decks all E(I) would, of course, be the same.

The P(I) can be calculated from the following tables and recursion formula.

	Single Deck	Double Deck	Four Deck	Infinite Deck
P(2):	$\frac{47}{49} \cdot \frac{46}{48}$	$\frac{95}{101} \cdot \frac{94}{100}$	$\frac{191}{205} \cdot \frac{190}{204}$	$\frac{12}{13} \cdot \frac{12}{13}$
R(I):	$\frac{(5-I)\cdot(48-I)}{(53-2I)\cdot(52-2I)}$	$\frac{(9-I)\cdot(96-I)}{(105-2I)\cdot(104-2I)}$	$\frac{(17-I)\cdot(192-I)}{(209-2I)\cdot(208-2I)}$	$\frac{12}{169}$

The factors R(I) reflect the probability of "opening" (drawing a new eight to a split eight) and "closing" (drawing a non-eight to an already split eight) the Ith split card. Our recursion formula is

$P(I) = N(I) \cdot R(I) \cdot P(I-1)/N(I-1)$ where the N(I) are magic numbers given by N(I) = 1,2,5,14,42,132,429,1430 for I = 2 to 9 respectively. To generate more of them, define F(I,0) = 1 and F(I,1) = I with

$$F(I,J) = \sum_{k=1}^{\min(I,J)} \binom{I}{k} F(2k,J-k) = F(I+1,J-1)+F(I-1,J)$$

so that F(I,J) is the number of distinguishably different ways that I items can "give birth" to J new ones in a branching process. Hence N(I) = F(2,I–2) = F(1,I–1).

To three decimals the P(I) are

I	Single Deck	Double Deck	Four Deck	Infinite Deck
2	.919	.884	.868	.852
3	.077	.102	.112	.121
4	.004	.013	.017	.021
5		.001	.003	.004
6				.001
Mean of I	2.085	2.132	2.156	2.182

If rules allow repeated splits up to a maximum of four, then P(4) = 1–P(2)–P(3).

This all assumes (X,X) is being split against Y ≠ X. Simple modifications can be made if it is (X,X) against X. For example, with a single deck, P(2) becomes

$$\frac{48}{49} \cdot \frac{47}{48} \text{ and } R(I) = (4-I)\cdot(49-I)/(53-2I)/(52-2I).$$

Dealer's Probabilities

Computers can determine the dealer's probabilities much more quickly with up-cards of nine and ten than with deuces, treys, and aces. The following table of the number of distinguishably different drawing sequences for one and many decks suggests the relative amount of computer time required.

	Ace	2	3	4	5	6	7	8	9	Ten
One Deck	5995	16390	10509	6359	3904	2255	1414	852	566	288
Many Decks	8497	18721	11125	6589	4024	2305	1441	865	577	289

A program can be written in BASIC in as few as 28 steps to cycle through all of the dealer's drawing sequences and weight the paths for a prescribed up card and deck composition. A reduction in time might be achieved by recognizing that, for example, a dealer total of 18 = 9 + 4 + 3 + 2 with a nine as up-card results from six distinguishably different sequences, namely the permutations of the three distinct denominations drawn, but from only one combination of these three cards. Programming complexities would arise, however, in treating 18 = 9 + 5 + A + 3 which could occur in only two sequences; the subroutine necessary to filter out the four forbidden orderings in the weighting process might well nullify the time advantage of such an approach based on *combinations* of cards with a particular total as opposed to *permutations* derived from the dealer's actual algorithm.

If initially we have A = number of cards in deck, T = J = up-card, W(I) = number of cards of denomination I, K = 0, B = 1, and F(X) = probability of achieving a total of X, then our program reads:

```
10 FOR I = 1 TO 10
20 IF W(I) = 0 THEN 270
30 B = B*W(I)/A
40 W(I) = W(I)-1
50 T = T+I
60 IF T>16 THEN 130
70 IF (T-11)*(T-7)>0 THEN 170
80 IF (I-1)*(J-1) = 0 THEN 150
90 FOR L = 1 TO K
100 IF A(L) = 1 THEN 150
110 NEXT L
120 GO TO 170
130 F(T) = F(T)+B
140 GO TO 240
150 F(T+10) = F(T+10)+B
160 GO TO 240
170 A = A-1
180 K = K+1
190 A(K) = I
200 GO SUB 10
210 I = A(K)
220 K = K-1
230 A = A+1
240 W(I) = W(I)+1
250 T = T-I
260 B = B*A/W(I)
270 NEXT I
280 RETURN
```

Distinguishably different Subsets

Several years ago Prof. E. O. Thorp counted the total number of distinguishable blackjack subsets of a single deck as $5^9 \cdot 17 = 33203125$. Since there are 2^{52} possible subsets there is an average duplication with respect to suit and ten denomination of about 130 million. The realization that there are only 1993 different subsets of size five was embarrassing to me, since I had simulated them 2550 times to test the validity of using the normal distribution approximation for the least squares linear estimators of deck favorability for varying basic strategy. The following table provides the number of subsets, both distinguishable and nondistinguishable, selected from a single deck:[A]

Size	# Dist.	#non-Dist	Size	# Dist.	#non-Dist
1	10	52	14	405350	$18\text{x}10^{11}$
2	55	1326	15	548090	45 "
3	220	22100	16	710675	$10\text{x}10^{12}$
4	715	270725	17	886399	22 "
5	1993	2598960	18	$10\text{x}10^5$	43 "
6	4915	$20\text{x}10^6$	19	12 "	76 "
7	10945	$13\text{x}10^7$	20	14 "	$13\text{x}10^{13}$
8	22330	75 "	21	15 "	19 "
9	42185	$37\text{x}10^8$	22	17 "	27 "
10	74396	$16\text{x}10^9$	23	18 "	35 "
11	123275	60 "	24	18 "	43 "
12	192950	$21\text{x}10^{10}$	25	19 "	48 "
13	286550	64 "	26	19 "	50 "

A program to construct the aforementioned 1993 as well as the 4915 distinguishable sets of size six made it possible to completely analyze hit-stand situations for variations in basic strategy. The results provide a worst case evaluation of the accuracy of the previously mentioned approximation since the interactions neglected by the linear estimates are most severe for small subsets and the normal approximation to their distribution is poorest at the beginning and end of the deck.

The $\binom{51}{5}$ and $\binom{51}{6}$ subsets that might be encountered for a given up card are achieved by weighting the distinguishable subsets properly. Favorability of hitting over standing was recorded for abstract totals of 12 through 17 (the exact composition of the totals remains unspecified since otherwise different weightings would have to be devised for the myriad ways the different totals could be realized) against all up cards except the ace. The dealer was assumed to stand on soft 17 and the few unresolved situations were completed by a formula which reasonably distributed the dealer's unfinished total on the shuffle up.

Actual frequencies of, and gain from, violating the basic strategy were recorded. The performances of Hi Opt I, Hi Opt II, and the Ten count were recorded in these situations.* Caution should be exercised in using these results for comparison at other levels of the deck since there is a pronounced effect of discreteness on card counting efficiency for small subsets in which only certain values of a system's parameter are realizable and those may not be particularly favorably located for the change in strategy contemplated.

The meaning of the following charts is best explained by example: With five cards left in the deck, perfect knowledge of when to hit hard 14 against a two is worth 16.4% to the player, who will exercise this option .518 of the time. (The conditional favorability of hitting in those situations where it is appropriate is 16.4%/.518 = 31%). In parentheses besides these figures appears the corresponding normal approximation estimate of potential gain (15.1%). The Hi Opt I, Hi Opt II, and Ten Count systems had respective efficiencies of 71, 77, and 68%. With six cards left in the deck precisely optimal hitting will occur .477 of the time, with a gain of 14.2% (estimated gain 13.4%), while the same three systems are 73, 79, and 66% efficient, respectively. The figures do not reflect the likelihood of the dealer having the given up card or the player possessing the particular total.

*Point values assigned to ace through ten are: Hi Opt I 0 0 1 1 1 1 0 0 0 -1; Hi Opt II 0 1 1 2 2 1 1 0 0 -2; Ten count 4 4 4 4 4 4 4 4 4 -9, although the Ten count is usually described by the ratio of non-tens to tens left in the deck.

GAINS FROM CHANGING HITTING AND STANDING STRATEGY WITH FIVE AND SIX CARDS LEFT

DEALER UP-CARD 2

	FIVE CARDS			SIX CARDS		
12	.381	14.7(13.2)	[71,72,85]	.374	12.7(11.7)	[54,65,74]
13	.563	17.4(16.4)	[64,72,78]	.555	15.3(14.7)	[67,74,73]
14	.518	16.4(15.1)	[71,77,68]	.477	14.2(13.4)	[73,79,66]
15	.445	14.8(13.3)	[81,72,58]	.400	12.4(11.5)	[84,74,60]
16	.365	12.8(10.9)	[65,69,44]	.325	11.2(9.2)	[67,68,51]
17	.184	4.8(3.4)	[10, - , -]	.143	3.3(2.5)	[21, - , -]

DEALER UP-CARD 3

	FIVE CARDS			SIX CARDS		
12	.418	15.7(14.0)	[82,83,89]	.413	13.8(12.5)	[82,85,82]
13	.549	16.4(16.2)	[66,76,78]	.554	14.4(14.5)	[66,78,74]
14	.503	15.3(14.9)	[72,79,64]	.468	13.3(13.0)	[66,80,63]
15	.441	14.1(13.4)	[81,71,47]	.391	11.8(11.5)	[69,72,52]
16	.349	12.6(11.5)	[62,66,33]	.320	10.1(9.6)	[55,66,37]
17	.173	4.9(3.9)	[09, - , -]	.140	3.4(2.9)	[16, - , -]

DEALER UP-CARD 4

	FIVE CARDS			SIX CARDS		
12	.562	15.7(14.7)	[68,79,95]	.548	13.9(13.3)	[72,81,89]
13	.492	14.8(14.5)	[71,79,79]	.502	12.8(12.8)	[75,82,78]
14	.469	14.0(14.2)	[75,79,59]	.450	12.0(12.2)	[76,76,61]
15	.426	13.1(12.8)	[81,68,36]	.371	10.8(10.9)	[82,69,44]
16	.342	12.0(11.1)	[58,61,32]	.311	9.4(9.3)	[57,62,27]
17	.175	5.0(4.3)	[07, - , -]	.140	3.4(3.2)	[16, -, -]

DEALER UP-CARD 5

	FIVE CARDS			SIX CARDS		
12	.533	14.2(13.3)	[67,80,96]	.516	12.4(11.9)	[72,82,92]
13	.465	12.9(12.7)	[70,81,79]	.462	11.0(11.1)	[74,82,79]
14	.422	11.9(12.5)	[75,79,55]	.394	10.1(10.7)	[76,80,59]
15	.386	11.7(12.2)	[76,66,35]	.356	9.5(10.2)	[70,66,34]
16	.319	11.1(10.7)	[46,54,30]	.303	8.5(8.8)	[53,57,25]
17	.206	6.6(5.6)	[22, -, -]	.184	4.6(4.2)	[24, - , -]

DEALER UP-CARD 6

12	.548	15.6(14.8)	[62,75,95]	.517	13.7(13.3)	[65,76,90]
13	.471	14.0(14.1)	[65,77,82]	.480	12.2(12.4)	[67,76,80]
14	.437	12.9(13.6)	[70,76,62]	.388	11.1(11.7)	[70,76,65]
15	.381	12.4(13.3)	[72,65,39]	.369	10.2(11.3)	[66,66,45]
16	.333	13.5(14.2)	[54,58,36]	.414	11.5(12.0)	[51,56,33]
17	.206	5.5(5.5)	[07, -,]	.169	3.8(4.1)	[19, -, -]

DEALER UP-CARD 7

12	.117	1.9(1.1)	[-, -, -]	.119	1.3(.8)	[-, -, -]
13	.263	4.6(3.9)	[-, -, 03]	.239	3.5(3.1)	[-, -, -]
14	.303	7.4(7.0)	[30,31,04]	.297	6.1(5.9)	[32,30, -]
15	.288	9.7(9.4)	[39,26,01]	.292	8.3(8.1)	[33,22, -]
16	.353	14.0(13.0)	[20,21,02]	.362	12.1(11.4)	[19,20,02]
17	.211	7.0(6.7)	[-, -, -]	.249	5.6(5.3)	[-, -, -]

DEALER UP-CARD 8

12	.218	4.3(3.7)	[-, -, 19]	.212	3.2(2.9)	[-, -, 03]
13	.213	3.9(2.9)	[-, -, 26]	.215	3.0(2.3)	[-, -, 18]
14	.287	4.9(4.8)	[21,29,17]	.278	3.9(3.9)	[20,23,09]
15	.317	7.0(7.5)	[58,34,14]	.309	6.0(6.4)	[55,28,14]
16	.333	10.8(10.7)	[32,27,11]	.350	9.4(9.3)	[31,27,14]
17	.437	11.5(10.9)	[-, -, -]	.395	9.8(9.5)	[-, -, -]

DEALER UP-CARD 9

12	.298	6.8(6.6)	[-, -, 29]	.305	5.5(5.5)	[-, -, 32]
13	.315	6.7(5.7)	[-, -, 27]	.314	5.4(4.8)	[-, -, 30]
14	.302	5.8(4.6)	[19,32,35]	.310	4.6(3.8)	[9,25,46]
15	.403	8.1(7.5)	[62,43,27]	.395	6.9(6.4)	[63,36,38]
16	.429	10.5(10.2)	[47,52,47]	.440	9.3(9.0)	[48,51,34]
17	.374	7.0(6.9)	[-, -, -]	.340	5.9(5.8)	[-, -, -]

DEALER UP-CARD T

12	.305	6.9(6.7)	[-, -, 33]	.314	5.7(5.7)	[-, -, 41]
13	.343	9.5(8.6)	[-, -, 26]	.361	7.8(7.4)	[-, -, 35]
14	.426	10.8(9.1)	[12,34,44]	.424	9.1(8.0)	[14,36,35]
15	.416	8.6(7.4)	[68,69,70]	.430	7.3(6.5)	[72,71,50]
16	.456	11.3(10.3)	[50,66,63]	.481	9.9(9.3)	[52,67,65]
17	.271	5.2(4.0)	[-, -, -]	.252	4.1(3.2)	[-, -, -]

That basic strategy was violated in more than half the cases for several situations is surprising. The greatest conditional gain in a hitting situation is the 40.5% for hitting hard 16 vs 6. The greatest conditional standing gain is almost 40%, with 16 vs 7. Hitting hard *17* is the most important variation against an eight and a system which counts A, 2, 3, 4 low and 6, 7, 8, 9 high would be nearly .900 efficient for making the play, although this is not recorded in the table.

Random Subsets stratified according to Ten Density

To examine the behavior of the normal approximation estimates for larger subsets, 3000 each of sizes 10 through 23 were simulated by controlling the number of tens in each subset to reflect actual probabilities. The only up-card considered was the ten because of the rapidity of resolution of the dealer's hand.

The effect of this stratification could thus be expected to be a reduction in the variance of the sample distributions proportional to the square of the Ten Count's correlation coefficients for the six situations examined. In addition to this reduction in variance of typically 40%, there would be the added bonus of saving computer time by not having to select the ten-valued cards using random numbers.

The results provide the continuum necessary to compare different card counting systems. Again, the following charts are best explained by example: with 10 cards left in the deck it was proper to stand with twelve in .269 of the sample cases. The gain over basic strategy was 3.11% in the sample, which compares with (3.16%) for the normal approximation. The Ten Count was 28% efficient, and a "special" system based on the density of the sevens, eights, and nines scored an impressive 78%.

The loss shown for the Ten counter playing a total of twelve with 21 cards left indicates the critical subsets with exactly 10 tens in them probably had an unduly large number of sevens, eights, and nines. A basic strategist (who always hits twelve) would have done better in this instance.

Hitting and Standing Gains Against a Ten

PLAYER TOTAL TWELVE

				T.C.	7-8-9 Special
10	.269	3.11	(3.16)	28	78
11	.244	2.49	(2.77)	17	74
12	.233	2.25	(2.43)	18	70
13	.204	1.96	(2.14)	10	64
14	.192	1.80	(1.89)	14	59
15	.177	1.37	(1.67)	11	46
16	.180	1.38	(1.47)	19	49
17	.170	1.12	(1.29)	05	35
18	.141	.93	(1.14)	09	64
19	.151	.99	(1.00)	06	72
20	.124	.82	(.89)	04	65
21	.112	.60	(.77)	–04(loss)	57
22	.106	.54	(.67)	04	53
23	.097	.51	(.58)	02	40

PLAYER TOTAL THIRTEEN

				T.C.	7-8 Special
10	.356	4.65	(4.46)	23	62
11	.321	4.08	(3.99)	23	56
12	.307	3.83	(3.59)	16	51
13	.283	3.23	(3.24)	22	45
14	.270	2.92	(2.93)	09	82
15	.254	2.45	(2.65)	16	78
16	.270	2.51	(2.40)	18	84
17	.267	2.18	(2.18)	09	78
18	.240	1.93	(1.97)	11	70
19	.249	1.98	(1.79)	14	72
20	.216	1.70	(1.61)	13	63
21	.189	1.37	(1.46)	05	60
22	.190	1.36	(1.31)	02	53
23	.179	1.17	(1.18)	07	49

PLAYER TOTAL FOURTEEN

				T.C.	H.O.I	H.O.II	H.O.I-7 Special
10	.383	5.68	(5.29)	42	12	40	90
11	.369	4.91	(4.82)	30	13	33	87
12	.368	4.51	(4.40)	33	10	27	86
13	.361	4.04	(4.04)	28	09	26	81
14	.356	3.73	(3.71)	29	04	27	88
15	.334	3.36	(3.42)	24	05	22	85
16	.332	3.20	(3.15)	24	09	28	86
17	.300	2.71	(2.91)	23	06	20	81
18	.304	2.68	(2.68)	25	03	20	80
19	.316	2.61	(2.48)	24	06	20	78
20	.289	2.40	(2.28)	22	04	21	77
21	.275	2.09	(2.11)	17	02	19	82
22	.272	1.99	(1.94)	16	01	20	80
23	.263	1.84	(1.79)	14	00	14	76

PLAYER TOTAL FIFTEEN

				T.C.	H.O.I	H.O.II
10	.418	4.51	(4.41)	59	75	66
11	.423	4.27	(4.06)	64	79	68
12	.414	3.79	(3.75)	53	79	67
13	.404	3.49	(3.47)	58	77	66
14	.397	3.26	(3.22)	63	81	71
15	.402	2.98	(3.00)	52	77	69
16	.390	2.80	(2.79)	57	78	70
17	.364	2.57	(2.60)	56	69	68
18	.371	2.46	(2.43)	53	69	65
19	.355	2.27	(2.27)	55	72	67
20	.351	2.25	(2.11)	58	72	68
21	.331	1.91	(1.97)	50	73	67
22	.337	1.86	(1.84)	55	71	68
23	.333	1.70	(1.72)	52	70	65

PLAYER TOTAL SIXTEEN

				T.C.	H.O.I	H.O. II	(H.O.I, 6-5) Special	"6-5"
10	.511	7.02	(6.93)	52	54	69	87	60
11	.519	6.51	(6.51)	58	55	70	90	60
12	.495	5.93	(6.13)	60	55	71	89	62
13	.523	5.93	(5.80)	51	57	72	90	61
14	.490	5.11	(5.50)	55	54	68	88	61
15	.534	5.14	(5.23)	58	56	72	89	60
16	.503	4.93	(4.97)	60	58	72	91	64
17	.506	4.73	(4.74)	55	57	71	88	60
18	.499	4.40	(4.53)	59	59	72	89	60
19	.501	4.43	(4.33)	61	59	71	91	65
20	.494	4.05	(4.14)	55	59	73	89	60
21	.492	3.99	(3.96)	57	57	70	90	63
22	.503	3.82	(3.79)	58	56	70	90	64
23	.500	3.79	(3.63)	54	58	74	91	64

PLAYER TOTAL SEVENTEEN

10	.175	1.83	(1.76)
11	.157	1.69	(1.50)
12	.150	1.48	(1.28)
13	.132	1.24	(1.09)
14	.131	1.03	(.93)
15	.121	.85	(.79)
16	.102	.73	(.67)
17	.091	.61	(.57)
18	.076	.46	(.48)
19	.076	.41	(.41)
20	.069	.36	(.34)
21	.063	.32	(.29)
22	.054	.24	(.24)
23	.047	.21	(.20)

The "special" system for playing totals of thirteen involved knowledge only of the remaining sevens and eights, the one for fourteen a combination of the Hi Opt I and the sevens, and that for sixteen an adjustment of the Hi Opt I count by twice the difference between the number of remaining sixes and fives. "Six-Five", the determination to stand solely on the basis of whether there remain more sixes than fives, was, by itself, more efficient for standing with sixteen than the Hi Opt I or Ten Count, scoring generally above 60%.

Extreme discontinuities in efficiencies as a function of the number of cards in the subset can usually be explained by one of the system's realizable values being very close to its critical change of strategy parameter. For example, the Ten Count's critical change ratio for standing with 15 is close to 2 others to 1 ten, and efficiencies take a noticeable dip with 12, 15, 18, and 21 cards in the deck. The Hi Opt I critical index for 15 is close to +1/17 or +1/18 and efficiency suffers correspondingly with 17 and 18 remaining cards. In such cases the card counting system, whether it suggests a change in strategy or not, is using up a considerable part of its probability distribution in very marginal situations.

Stratified Sampling used to analyze Expectation in a particular Deck

The following approximate computations show that the variance of a blackjack hand result is about 1.26 squared units.

Player Result	Approximate Probability	Squared Result	
± 2	.10	4	Average Squared Result = 1.26
$+1.5$	.05	2.25	
± 1	.75	1	
0	.10	0	

More refined calculations will not change this average squared result appreciably and it will be the same as the variance since the square of the average result may be assumed to be effectively zero.

Suppose a sample of thirteen independent blackjack hands is simulated. The sample sum will have a variance of 13(1.26). It is worthwhile to study the consequences of stratified, rather than random, sampling.

Let the thirteen hands now be played against each of the denominations ace through king as dealer up-card. Then the variance of the sum would obey

$$\operatorname{Var} \sum_{i=1}^{13} X_i = \sum_{i=1}^{13} \operatorname{Var} X_i = \sum_{i=1}^{13} (EX_i^2 - (EX_i)^2) =$$

$$13(1.26) - \sum_{i=1}^{13} (EX_i)^2,$$

the last equality resulting not because *each* EX_i^2 for dealer up-card i is 1.26, but rather because their average is 1.26.

Thus the average variance for these stratified sample observations has been reduced from 1.26 to

$$1.26 - \sum_{i=1}^{13} (EX_i)^2/13.$$

The subtracted term, which provides the variance reduction, is the averaged squared expectation for blackjack hands when the dealer's up-card is known. Using Epstein's tables of player expectation as a function of dealer up-card, we find this average square to be .04, which, by itself, provides only a modest reduction in variance to 1.22.

The same principle, albeit with more elaborate symbolism, can be used to show that controlling the player's first card as well as the dealer's up-card will reduce variance by .06, this figure again being derived from Epstein's tables. The average squared expectation for three card situations, where player's hand and dealer's up card are specified, is .24, the first really significant reduction achieved by this sort of stratification.

To get an approximation to how much variance reduction would result if four or more cards were forced to obey exact probability laws in the sample, we can assume the resolved hands have the same 1.26 average squared result while the unresolved ones may be assigned the three card figure of .24, although this almost surely is an underestimate. Then, employing some of Gwynn's computer results which show that about 17% of all hands require four cards, 40% five cards, 28% six cards, 11% seven cards, and 4% eight or more cards, we complete the following table:

Number of Cards Controlled Precisely	Average Squared Expectation	Variance
0	.00	1.26
1	.04	1.22
2	.06	1.20
3	.24	1.02
4	.42	.84
5	.83	.43
6	1.11	.15
7	1.22	.04
8	1.26	.00

The immediately evident benefit of this is reduction of sample size necessary to produce a desired degree of statistical accuracy. Beyond this, however, lurk even greater savings in computer time since the number of cards actually simulated with random numbers would be very few.

Suppose, for example, that one investigated the player expectation in a 30 card deck by sampling 8550360 hands to reflect the $\binom{30}{2} = 435$ possible player hands, the dealer's $28 \times 27 = 756$ up and down cards, and the 26 possible first hit cards. This would have the same variance as a purely random sample of about 25 million hands.

Moreover, only about 5 million cards would have to be generated to complete the 8.55 million hands, whereas the 25 million independent hands would use an average of 5.4 cards each, or a total of about 135 million. A computer subroutine to cycle through and weight the 55000 distinguishably different five card situations would add very little running time to the generation of the 5 million randomly drawn cards.

Use of Infinite Deck Approximations

An "infinite deck" is, of course, not really infinite at all. Infinite deck approximations are calculations based on the assumption that removal of the cards already in the hand has no effect on the probability of their subsequent occurrence. The terminology results from the fact that the larger the deck is initially, the smaller will be the error occasioned by sampling with replacement, it diminishing entirely in the limit.

The appeal of the infinite deck is the speed with which it can be analyzed on the computer. Exact analysis of the player's complete infinite deck expectation takes only a trice, whereas the Manson group's 4-deck program took about an hour on a very fast computer and it was still inexact in some details.

To communicate a feeling for the magnitude of error involved, as well as to suggest methods of refining and improving approximations, several case studies will be presented. Exact expectations are taken from Epstein's book for the single deck case and from the Manson paper for four deck situations.

Complete Infinite Deck expectations are calculated assuming every denomination has 1/13 chance of occurring independently of its previous appearance or non-appearance. 49 Card Infinite Deck figures are based on the same assumption of independence, but with the player's initial cards and the dealer up card removed, so that probabilities are integral fractions with 49 as a denominator. 48 Card Infinite Deck calculations take further account of the player's first hit card or the dealer's down card. Similarly we have 205 and 204 Card Infinite Deck expectations in the four deck example.

SINGLE DECK EXPECTATIONS

Player Hand	Dealer Up Card	Exact	48 Card Infinite Deck	49 Card Infinite Deck	Complete Infinite Deck
(T,T)	9	.7440	.7439	.7431	.7584
(T,6)	9	-.4793	-.4793	-.4807	-.5093
(7,6)	9	-.4185	-.4188	-.4172	-.3872
(7,3)	9	.1537	.1464	.1603	.1443
(5,3)	9	-.2171	-.2173	-.2249	-.2102
(T,T)	4	.6448	.6450	.6475	.6611
(T,6)	4	-.1935	-.1964	-.1914	-.2111
(7,6)	4	-.1584	-.1611	-.1559	-.2111
(7,3)	4	.5704	.5760	.5652	.4609
(5,3)	4	.0866	.0883	.0860	.0388

FOUR DECK EXPECTATIONS

Player Hand	Dealer Up Card	Exact	204 Card Infinite Deck	205 Card Infinite Deck	Complete Infinite Deck
(T,T)	9	.7549	.7549	.7547	.7584
(T,6)	9	-.5021	-.5021	-.5024	-.5093
(7,6)	9	-.3946	-.3947	-.3944	-.3872
(7,3)	9	.1465	.1450	.1482	.1443
(5,3)	9	-.2120	-.2121	-.2135	-.2102
(T,T)	4	.6571	.6572	.6578	.6611
(T,6)	4	-.2071	-.2078	-.2065	-.2111
(7,6)	4	-.1988	-.1995	-.1982	-.2111
(7,3)	4	.4872	.4885	.4859	.4609
(5,3)	4	.0502	.0506	.0501	.0388

Generally the approximations are better for large up cards like the 9 than for small ones like the 4. The greatest inaccuracies seem to occur for doubling down since the error in expectation is doubled. All this is understandable and there may be some way to combine exact, without replacement, methods and the infinite deck, with replacement, techniques to produce a satisfactory trade off of accuracy for speed.

Cascading Process for Determination of Best Strategy

The fact that there are 3082 ways to create integral totals not exceeding 21, using no terms in the sum greater than 10, was exploited by the Manson group in their treatment of four deck blackjack. They were able to determine absolutely exact multiple card strategies and expectations without calculating the dealer's probabilities more than once for any possible player hand and dealer up card.

The process begins with the formation of all possible player totals of 21, such as (T,T,A), (T,9,2), (T,9,A,A), . . . (2,2,2,A, . . . A). For each of these player hands, the dealer's exact probabilities are figured for the up card being considered and from this the player's standing expectation is computed, stored, and indexed for retrieval. (The indexing and retrieval mechanism for all possible hands is one of the more difficult aspects of the computer program.)

Next, all possible player totals of 20 are formed, starting with (T, T). Now (playing devil's advocate at this stage) one calculates the player's expectation from hitting the total of 20 by referring to the standing expectations already catalogued for the hands of 21 which might be reached if an ace were drawn. Then this hitting expectation is compared to the player's standing expectation with the currently possessed total of 20, which is calculated as in the previous paragraph.

In this manner the computer cycles downward through the player's totals until finally the exactly correct strategy and expectation is available for any possible player hand. The procedure is not, of course, restricted to four deck analysis; applied to any prespecified set of cards it will yield the absolutely correct *composition* dependent strategy and associated expectation, without any preliminary guesswork as to what *totals* the player should stand with.

The cascading process can also be harnessed together with the pair splitting algorithm in the early part of this chapter to answer, once and for all, questions about what is the best strategy and consequent expectation for any number of decks and any set of rules. A summary of strategies, including two card "composition" dependent exceptions, follows:

BASIC STRATEGY FOR ANY NUMBER OF DECKS

Outline Key:

A. Dealer stands on soft 17

B. Dealer hits soft 17

C. Exceptions when double after split permitted.

I Single Deck

A. Follow pages 18 and 20

B. Modify IA: hit soft 18 v A and stand with (T,2) v 6

C. Split (2,2) v 2, (3,3) v 2,3, and 8, (4,4) v 4-6, (6,6) v 7, and (7,7) v 8; also split (9,9) v A if dealer hits soft 17

II Two Decks

A. Follow page 18, except: don't double 8 v 5-6, (2,9) and (3,8) v A, (A,6) v 2, (A,2) and (A,3) v 4, and (A,8) v 6; don't split (2,2) v 3; hit (A,7) v A and (T,2) v 4

B. Modify IIA: double 11 v A, (A,7) v 2, (A,3) v 4, and (A,8) v 6; hit soft 18 v A; stand with (8,4) and (7,5) v 3

C. Split as in IC except don't split (4,4) v 4, (3,3) v 8, and (9,9) v A

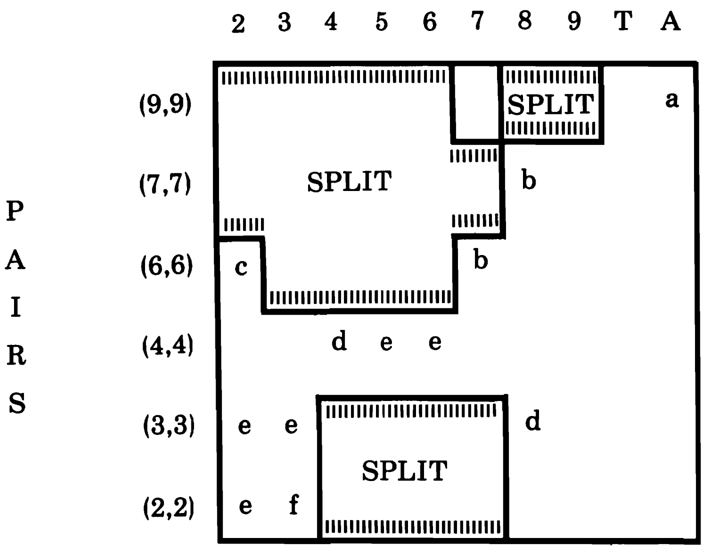
III Three Deck exceptions to II

A. Don't double (3,6) and (4,5) v 2 and 11 v A; don't split (6,6) v 2; hit soft 18 v A

B. Don't double (3,6) and (4,5) v 2 and (A,3) v 4; don't split (6,6) v 2; stand with (T,2) v 4; hit (8,4) and (7,5) v 3

C. Split as in IIC except don't split (6,6) v 7 and (7,7) v 8

IV Four Deck exceptions to IIIA and B: don't double 9 v 2


2 3 4 5 6 7 8 9 T A
P A I R S
(9,9)
(7,7)
(6,6)
(4,4)
(3,3)
(2,2)
SPLIT
SPLIT
SPLIT
a
b
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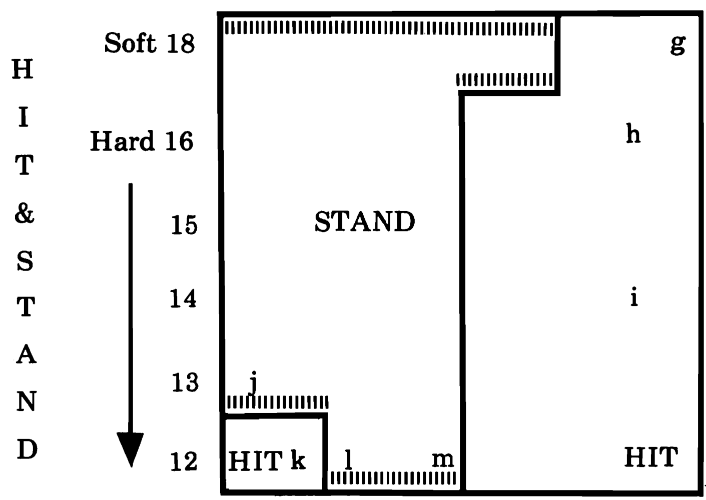
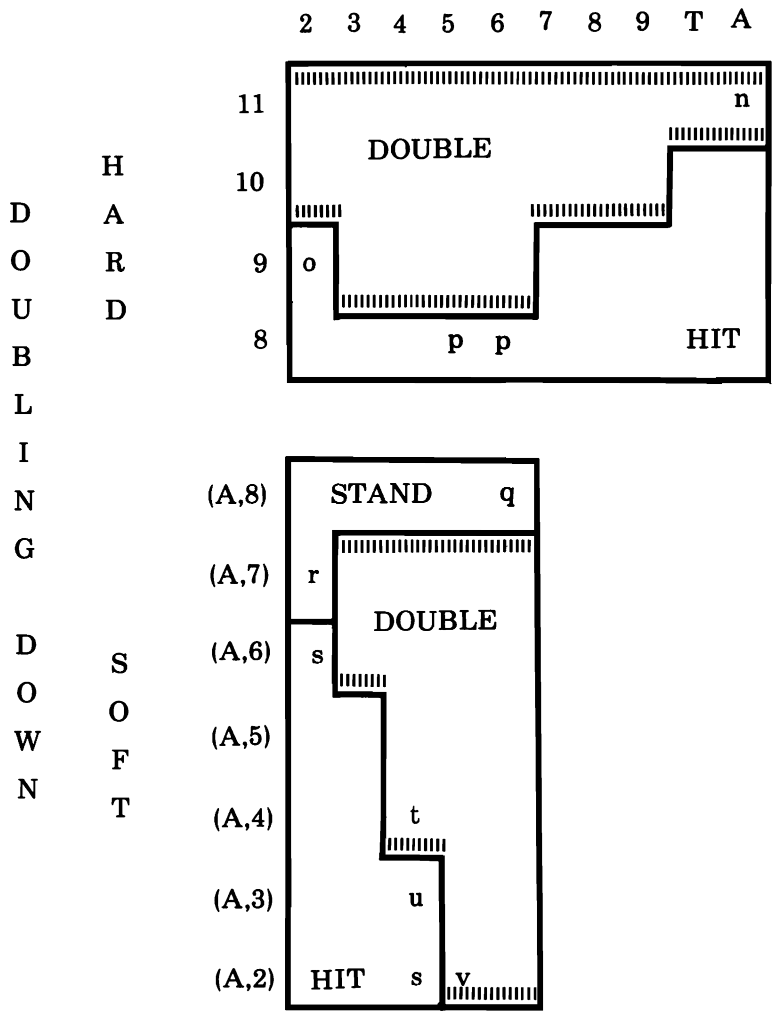
d e e
e e
d
e f


H I T & S T A N D
Soft 18
Hard 16
15
14
13
12
STAND
g
h
i
j
HIT k
l
m
HIT



Always split (A,A) and (8,8), hit soft 17 and below, and stand on soft 19 and above.

VIII Eight Deck exception to IVA: stand with (T,2) v 4

IX Nine Deck exception to VIIIA: don't double (A,2) v 5

X Ten Deck exception to IVB: don't double (A,2) v 5

XXVII Twenty-seven Deck exception to XA: don't double (A,4) v 4

The strategy for any number of decks not listed is found by referring to the next lower number which is; for example, the six deck strategy is identical to the four deck one.

Those for whom charts are more enlightening will find the same information displayed graphically on the last two pages. Small letters in the charts reference the following explanations within which "s17" denotes "dealer stands on soft 17," "h17" means "dealer hits soft 17," and "das" stands for "double after split."

a. Split in one deck if both das and h17
b. Split in one and two decks if das.
c. Split in one and two decks or whenever das.
d. Split in one deck if das.
e. Split whenever das.
f. Split in one deck or whenever das.
g. If s17 then stand in one and two decks, except hit (A,7) in two decks.
h. Stand on three or more cards in fourteen or fewer decks.
i. Stand with (7,7) in one deck
j. Hit (T,3) in one deck
k. Stand with (8,4) and (7,5) in one deck and also stand with these hands in two decks if h17.
l. Hit (T,2) in one and two decks; hit in seven or fewer decks if s17.
m. Hit (T,2) in one deck if s17.
n. If s17 don't double in three or more decks and don't double (2,9) and (3,8) in two decks.
o. Double in one and two decks and also with (7,2) in three decks.

p. Double (5,3) and (4,4) in one deck.
q. Double if h17; double in one deck if s17.
r. Double in two or more decks if h17.
s. Double in one deck only.
t. Don't double in twenty-seven or more decks if s17.
u. Double in one deck; double in two decks if h17.
v. Don't double in ten or more decks; don't double in nine decks if s17.

Instruction h., to stand with multiple card totals of 16 v T, is not given in the outline form, which includes only two card exceptions to the optimal total dependent strategy. Employing this suggestion, along with the other two card exceptions to total dependent strategy, enables the basic strategist to play an almost optimal composition dependent basic strategy. A slightly more complicated but effective rule in this case would be to stand on all multiple card totals of 16 which don't contain a six.

The player's expectations from using an exact, composition dependent, basic strategy appear in the following chart, where the parenthetical figure is the expectation when the dealer hits soft 17.*

BASIC STRATEGY EXPECTATION (IN%)

Number of Decks	No Double After Split		Double After Split	
Infinite	−.65	(−.87)	−.51	(−.73)
Four	−.49	(−.70)	−.35	(−.55)
One	.04	(−.15)	.18	(−.00)
Half (26 cards)	.87	(.70)	.99	(.83)
Quarter (13 cards)	3.09	(2.97)	3.09	(2.97)

*Not all references, charts, etc. from the first edition have been revised to reflect these recent findings.

As mentioned in Chapter Two, a "total dependent" basic strategy will yield an expectation slightly lower than these figures, .04% less for a single deck. The method of interpolation by reciprocals suggests that the player would lose about .01% from applying a total dependent strategy in a four deck game. In an infinite deck there is no distinction between total dependent and composition dependent strategies.[B]

In casinos where the surrender rule prevails those who do not live by the warrior's code and are willing to raise the white flag should be guided by the following chart. See page 123 for adjustments to the basic strategy player's overall expectation.

Number of Decks for Basic Strategy Surrender

Player's Hand	Dealer's Up Card 9	T	A(s17)	A(h17)
(7,7)		1 only		1 only
(8,7)		7 or more		4 or more
(9,6) and (T,5)		All		All
(8,8)				2 or more
(8,8) das				3 or more
(9,7)	3 or more	All	2 or more	All
(T,6)	4 or more	All	All	All
(9,8)				2 or more
(T,7)				All

APPENDIX TO CHAPTER 11

A.

My method of counting the distinguishable subsets of size n was to cycle through the number of tens, t, in the subset. Then I used a recursive generation of the number of subsets of size n—t which could be formed using only the nine non-ten denominations.

Thorp recommends a more elegant technique. $\binom{10+n-1}{n}$ is the number of ways to assign n indistinguishable marshmallows to 10 distinguishable raccoons. We can think of the number of marshmallows given to the jth raccoon as being equivalent to the number of times that the jth card denomination is repeated in our n card subset. He then corrects by subtracting the number of subsets which are forbidden by restriction on the different denominations available. For example, with $n=5$, he gets a preliminary $\binom{10+5-1}{5}$ = 2002 and then subtracts the 9 subsets which consist of five cards of one of the lower denominations, A,2,3,---,or 9, to get the required 1993.

B.

The expectations themselves appear to support interpolation by reciprocals as described at the beginning of Chapter Eight, but a plausible refinement of the method gives much better results. When the four deck player makes his first playing decision he will have seen at least three cards; hence the uncertainty he confronts will consist of the other, at most 205, cards, perhaps 204.5 on the average. The reciprocals of 204.5, 48.5, 22.5, and 9.5 for four, one, half, and quarter decks align startlingly well with the corresponding basic strategy expectations.

12

UPDATE—FIFTH NATIONAL CONFERENCE ON GAMBLING

"Let us swallow them up alive as the grave;
and whole as those that go down into the pit:

We shall find all precious substance,
we shall fill our houses with spoil:

Cast in thy lot among us;
let us all have one purse:"
Proverbs 1:12-14

Those thirsting for knowledge of the financial, social, or mathematical aspects of gambling owe a large debt to Professor Bill Eadington for his tireless efforts in organizing the biennial national conferences on gambling sponsored by the University of Nevada at Reno. These gatherings have provided a clearing house for information and a meeting place for a variety of people, from gaming and law-enforcement officials to academicians of all stripes. I suspect that had it not been for the opportunity afforded me at the First Annual Conference at the Sahara Hotel in Las Vegas in June of 1974 I would never have achieved an audience for, and recognition of, my work.

In October of 1981 the Fifth National Conference on Gambling was held at Caesars Tahoe on the lake of the same name. I shall use this final chapter to bring the book up to date with a record of my participation. Who knows, perhaps future conferences will serve as a metronome for subsequent revisions.

Improving Strategy against the Dealer's Ace

In the summer of 1980 Paul Bernhardt of the Stanford Electronics Lab informed me that the effects of removal for some strategic situations, tabulated on pages 74-85 of the first edition, did not sum to zero as the theory underlying their derivation and use suggested they should. My initial, defensive, reaction was that discrepancies were due to either round-off errors, misprints, or a somewhat approximate dealer's drawing technique.

Since by that time I had evolved a rapid and exact dealer's drawing algorithm I set to work recalculating them more precisely for the second printing of the book. Nevertheless, try as I would, I could never get them to add up to zero when the dealer's up card was an ace. It slowly dawned on me that the great blackjack god in the sky had never intended them to and that the reason was that all cards were *not* equally likely to have been removed from the deck *if the dealer did not have a blackjack under the ace.*

This Bayesian realization necessitated a different set of formulas to produce least square estimates of strategic favorability when the dealer's up card was an ace. Out of a deep humanitarian concern for the reader I shall relegate all technical discussion to the appendices and concentrate here on the practical implications to the card counter. Suffice it to say that the algebraic agony of deriving these formulas and the complexity of their application have convinced me that the casinos should change the rules of blackjack: if the dealer did not look at his hole card for a board sweeping blackjack, but, rather, allowed the player to tie his natural 21, the mathematical travail would be greatly diminished!

For card counters, the major importance of the formulas derived in Appendix A is the fact that different change of strategy parameters will be required at different levels of the deck when the dealer's up card is an ace. To develop some insight into why this should be so, imagine you're contemplating whether to hit a total of 12 against an ace with a two card remainder you know contains exactly one ten. Your chance of busting is not 1/2 as you might imagine: you're certain to bust since the dealer's failure to turn over a blackjack means that he *doesn't* have the ten underneath and you *must* get it if you draw. [A]

We shall examine this matter using that old war-horse, Thorp's Ten Count, as our vehicle. The following chart displays the critical ratio of other cards to remaining tens which makes drawing and standing equal in expectation at various levels of the deck.

Critical Ten Count for Standing against an Ace

Player's Total	Cards remaining in deck				
	50	40	30	20	10
17	3.10	3.12	3.16	3.23	3.47
16	1.498	1.51	1.53	1.58	1.72
15	1.38	1.39	1.41	1.45	1.59
14	1.26	1.27	1.29	1.33	1.46
13	1.16	1.17	1.19	1.22	1.35
12	1.05	1.06	1.08	1.12	1.23

The figures with 50 cards left in the deck are reasonably consistent with Thorp's published Ten Count strategy, although his figures were not given with a sufficient number of digits for precise comparison. Since his method of analyzing rich decks was to add tens to a full deck, some of his interpolated standing ratios may well have been lower before being rounded off.

The revised theory also suggests that the player should become more aggressive than existing count strategies suggest in doubling down against the ace. Naturally these findings apply as well to games where the dealer hits soft 17.

Card counting systems other than the Ten Count will also have progressively less extreme indices for standing and doubling against the ace. The appropriate changes in indices as the deck is depleted will not be as dramatic, but proper analysis will nevertheless reveal increasing opportunity to the card counter: for instance, with less than half a deck remaining the Hi Opt I player should stand with 16 for any positive count when the dealer hits soft 17.

A Digression on Precise Pinpointing of Strategic Indices

In 1967 Epstein wrote: "Computation of decision strategies . . . is not straightforward. Since a large number of card configurations can correspond to a particular count, either an 'average' or a 'most probable' configuration must be used to ascertain the strategy attendant to that count. To determine the former is not mathematically tractable, while the latter may not be representative." On page 99 of this book I attempted to share Epstein's insights.

As a simple example of the problem involved, imagine that a ten counter has an abstract total of 15 against the dealer's equally phantom ace. Should the player draw or stand, knowing that there are exactly six tens and nine non-tens remaining?

If the denominations of the nine non-tens were known, the question could be answered with exactitude by setting up the appropriate 15 card deck for the computer to analyze. However, a ten counter has no such specific knowledge of the non-tens, knowing only that they are nine in number. One attempt at solution is to propose Epstein's "most probable configuration" of precisely one card of each denomination and six tens. For this subset the player's expectation by drawing is −.5539, while standing yields −.5391. Thus, at this level of analysis, standing appears preferable.

Note that this allows no possibility for the dealer to draw a five to a down card of five and yet this could really happen in such a 15 card subset about which all is known is that it contains six tens. The realization dawns that what is required here is the average gain by drawing for each of the $\binom{36}{9} =$ 94,143,280 equally likely subsets which the player might be confronting.

The knowledge that only 19855 of these subsets are distinguishably different provides little comfort and encouragement here. The task appears herculean and its magnitude has hitherto dissuaded analysts from seeking *the* answer in cases like this.

And yet the problem can be solved, and in little more computer time than was necessary to produce an answer for the ideal, "average" 15 card deck with six tens. The secret is to allow the computer to range through all possible hands of a full 52 card deck, but to probabilize them subject to the card counting information that there are exactly six tens and nine nontens available initially.

Thus the chance of the dealer having a five as down card and then drawing another five to it is $1/9 \times 8/14 \times 3/35$. Similarly the dealer's chance of ending up with (A,2,2,2) becomes

$$1/9 \times 8/14 \times 3/35 \times 7/13 \times 2/34.$$

I have programmed the computer to do just this sort of reprobabilizing of blackjack hands to reflect the alterations in likelihood associated with knowledge of a card counting system's parameters. The exact solution to our 15 card, six ten subset is a bit surprising: *drawing* is preferable to *standing* by −.5491 to −.5502. It is this sort of procedure which was used to verify the shifts in critical Ten Count standing numbers in the previous table. Such a method can also be used to analyze other point count systems, although the probabilistic subroutine is a bit more complicated. [B]

When Reshuffling is necessary to finish a Hand

"He balanced fives against tens"
Carl Sandburg

At the beginning of Chapter Three it is argued that the expectation on the second hand dealt from a single deck must be the same as that on the first hand because we are guaranteed that the second hand always can be finished before the pack runs out. But what about subsequent hands which may or may not require the pack to be reshuffled in order to finish them? Will their expectation (using, of course, the same full deck basic strategy) be the same as the first hand's? Off hand it seems possible, if not likely, that this would be the case, even though the means of proof used in Chapter Three is no longer available.

I've discovered that a useful method to study blackjack questions of this type, which involve dealing patterns, is to contrive small decks of cards for which the number of possibilities is kept manageable. Then an exhaustive analysis may suggest the true state of affairs and, possibly, a direction of proof if one exists. Toward this end, let us imagine a "deck" consists of nine cards of which three are fives and six are tens. Our game will be ordinary blackjack with the exception of forbidding pair splits.

Elementary calculations like those on page 22 show that, for the full nine card deck, the player's best strategy is to double (5,5) v 5 and otherwise stand with all totals of 15 or higher. The corresponding "basic strategy" (for so we shall refer to it) expectation is 5.95%.

Now, the total number of pips in our pack of cards is 75 so if we use up 35 on the first hand, there will be enough (40) to guarantee that the second hand is finished whereas if we use up 45 pips on the first round (our 20, dealer's 25) the 30 remaining will not allow the hand to be finished without reshuffling. This, then, is just the sort of example we need in order to gain insight into what happens when reshuffling may or may not be necessary to finish an ordinary blackjack hand.

The number of possibilities is sufficiently small here (unlike real blackjack from a 52 card deck) for us to establish precisely the link between the cards on the first and second hands. What we discover by pursuing all the possibilities is a bit surprising to say the least. [C] The player's advantage on the second hand, assuming he adheres strictly to the full deck basic strategy, has dropped to .36%!

One suspects there must be some sort of restorative force gravitating toward the full deck advantage, so let's look at a third hand, begun with whatever is left for play after the second hand is finished (this sort of continuous dealing was used until recently at the Nevada Club in Reno — the play itself decided when the dealer reshuffled). True enough the player's advantage rises on the third hand, but only to 4.48%, not as high as for the full deck. Subsequent hands in this continuous dealing process have the following expectations:

Hand Number	Expectation (%)
1	5.95
2	.36
3	4.48
4	.62
5	3.66
6	1.30
7	3.08
8	1.74
9	2.74
10	1.99
11	2.55
12	2.13
30 and all thereafter	2.31

So the player's long run expectation, with this mode of continuous dealing, is 2.31%, less than half the expectation for the full deck!

Certainly few, if any, people play blackjack from our nine card deck ("5 & 10" or "Woolworth" blackjack we might call it!). Nevertheless the example is valuable because it demonstrates the futility of casting about for some alternative proof that the third and subsequent hands from a normal blackjack deck would have the same expectation as the first and second. It's almost certain that they do not, although the actual alteration in expectation is unlikely to be as dramatic as what we have observed in our contrived small deck.

Percentage Advantage from Proportional Betting Schemes

One of the most interesting submissions to the Fifth National Conference was a paper entitled "An Analytic Derivation of Win Rate at Blackjack" and presented by Gary Gottlieb from New York University. Gottlieb used a special random walk model called "Brownian Bridge" (I must confess that my first hearing of the term conjured up images of a whist-like game played by elves!) to study the profit rate of a player who only bets on favorable decks. Under certain generally reasonable conditions he derived a single expression (in terms of fearsome arcsines, square roots, and logarithms) which measured how much the player would bet and could win if he sat out all unfavorable hands and bet a fixed fraction of his capital in proportion to his advantage otherwise.

We can call upon the normal curve and UNLLI from Chapter Six to produce the same quantifications, and for practically any circumstances. As an example, suppose one is playing four deck blackjack with standard downtown Las Vegas rules (dealer hits soft 17). He has a perfect knowledge of what the instantaneous basic strategy expectation is, a \$10,000 bankroll, and bets his perceived advantage times his bankroll when, and only when, he has an edge. What will his average bet and average earning be for wagers made with exactly 130 cards remaining to play?

To answer this question we need several parameters as a preliminary:

$m = -.70$ (%) is the full deck expectation (see Chapter Eleven)

$ss = 2.84$ from page 71

$N = 208$ (four decks) and $n = 130$

$$b = 51 \sqrt{\frac{ss\,(N-n)}{13\,(N-1)n}} = 1.28$$

$$z = \frac{0-m}{b} = \frac{.70}{1.28} = .54$$

.1857 from the UNLLI chart on page 87

$.1857 \times b = .2377$(%) which will be the average perceived advantage

Hence the average bet at this level (remember, many of the bets are zero, when the deck is bad) will be .2377 per cent of \$10,000, or \$23.77.

To find the average earning we must first determine how often the deck is favorable. On page 91 we find the normal curve area to the right of $z = .54$ is $.5000 - .2054 = .2946$ which is also the fraction of the 130 card remainders that are favorable to the player. (Note that the average bet, *when one is made*, is \$23.77/.2946 = \$80.69, although this number plays no role in our calculations.) The average earning will be given by the formula

$$m \times \text{average perceived advantage} + b^2 \times \text{probability of favorable deck}$$

Plugging in our figures we crank out an average profit of $-.70 \times .2377 + (1.28)^2 \times .2946 = .3163$ which is in percent of percent of our bankroll. Thus our average earning per hand is \$.32. Our percentage advantage on money invested would be .32/23.77 = 1.33%.

What is a trifle unrealistic here is the notion that the player can diagnose his advantage perfectly. If a card counting system with betting correlation $\rho = .96$ were used, we would multiply the original value of b by ρ and get a revised $b = 1.28 \times .96 = 1.23$ and repeat the calculations, getting:

$$z = \frac{.70}{1.23} = .57$$

.1770 from page 87

$.1770 \times 1.23 = .2177(\%)$, giving an average bet of \$21.77

Area to the right of $z = .57$ from page 91 is $.5000 - .2157 = .2843$

Average earning of $-.70 \times .2177 + (1.23)^2 \times .2843 = .2777$ (% of %), or \$.28 with a percentage advantage on money bet of .28/21.77 = 1.28%

Naturally, to assess total performance throughout the shoe one would repeat these calculations for various values of n one expected to encounter, and not just for n = 130.

Games Which Have An Advantage for the Full Deck

When the player has a basic strategy advantage for the full pack then the previous techniques must be modified slightly in the determination of average perceived advantage and probability of encountering a favorable deck. As an illustrative example we'll investigate performance at what I regard currently as the best game in the world, single deck blackjack at

Caesars Palace in Las Vegas. The right to double after split and surrender raises the basic strategist's advantage to m = .20(%), so our $10,000 bankroll player wants to bet $20 on the first round; perfectionists try to bet $20.48 but are consistently rebuffed by the dealers who then enforce the table minimum of $25, thus costing the house $.05 per hand instead of $.04.

Suppose all six spots are in use so the second and last bet is essayed with about n = 33 cards left from the pack of N = 52. We calculate

$$b = 51 \sqrt{\frac{2.84\ (19)}{13\ (51)(33)}} = 2.53$$

$$z = \frac{0-m}{b} = \frac{-.20}{2.53} = -.08$$

Now, we must ignore the algebraic sign and look up the UNLLI figure for z = .08, which is .3602.

This, as previously, is multiplied by b to produce .3602 × 2.53 = .9113. When m is positive (and z is negative) we must add m itself to this figure to produce the average perceived advantage, which is .9113 + .2048 = 1.1161(%), indicating an average bet of $111.61.

To find the probability of encountering a favorable deck we take the area to the right of z = −.08. This area is found by *adding* the tabulated area for z = .08 to .5000. Thus, from page 91, we get .5000 + .0319 = .5319 as the frequency of favorable hands with 33 cards remaining. Now, the average earning is a piece of cake:

$$m \times \text{a.p.a.} + b^2 \times \text{p.o.f.d.} = .20 \times 1.1161 + (2.53)^2 \times .5319 = 3.6278(\%\ \text{of}\ \%)$$

Hence our perfectly perceiving proportional punter earns an average of $3.63 on the second hand dealt with 33 cards remaining and played with basic strategy.[D]

We'll conclude with what must be regarded by the advocates of proportional wagering as a paradox, namely an example showing that a game with poorer expectation can produce a higher return on money invested even if the deck is dealt to the same level. The single deck game at the El Cortez Hotel in downtown Las Vegas has the same rules as Caesars Palace except the dealer hits soft 17, resulting in a net expectation of m = .03(%).

Using the same assumption and technique as in the previous example, we find that the El Cortez player with a $10,000 bankroll bets $3.00 on the first hand and an average of $102.81 on the second hand. His average earning on the first hand is less than a tenth of a cent while on the second hand it is $3.26. So the total percentage return on the money bet in one deck is 3.26/105.81 = 3.08%. This is to be contrasted with the overall figure of 3.67/131.61 = 2.79% for Caesars Palace.

Naturally more money is won at the better game, but the percentage return, in this case, turned out to be smaller because more money was bet early in the deck on a rather small advantage. The same sort of situation can develop in comparing two multiple deck games dealt down to the same level: deep down in the deck the game with poorer full deck expectation may begin to show a higher cumulative percentage yield, albeit producing less total revenue.

Final Thoughts

A couple of weeks before this second edition was put to bed I received an inquiry from some aspiring computer-blackjack men in Reno. Their interest was in how they might better estimate the player's predeal advantage to take into account, among other things, how the chance of being dealt a blackjack fluctuates.

My first impulse was to suggest that what they were after was what a statistician would call an "interactive" model for estimating advantage, based on different weighting factors for each possible *pair* of cards remaining in the deck. I offered the opinion that this would not be a fruitful avenue to pursue because of both the difficulty in calculating the many new parameters and the unlikeliness of significant improvement over existing 10 term linear estimation.

However, as I gave more thought to the matter I realized that already embedded within the conventional effects of removal on player advantage for basic strategy are two separate, independent, and additive components. One component is the effect of removal on the player's 3 to 2 bonus for an uncontested "natural," or blackjack, and the other is the effect of removal on all the other aspects of blackjack (which we could call the "unnatural" aspects).

And then it struck me. Why should we *estimate* something which is trivial to calculate exactly? (Remember the people who contacted me were building a computer.) The player's expectation from natural blackjack bonuses is given by

$$a \cdot t \cdot \big(1-2(a-1) \cdot (t-1)/(n-2)/(n-3)\big)/n/(n-1)$$

where a, t, and n are the numbers of remaining aces, tens, and cards respectively. For instance, for a full 52 card deck we get

$$4{\cdot}16{\cdot}(1-2{\cdot}3{\cdot}15/50/49)/52/51 = .0232.$$

Now this is by no means an oracular revelation (Wilson has almost the same calculation in his book), but what it suggests is that, by uncoupling the blackjack bonus from the "unnatural" aspects of the game, we can calculate the former exactly and estimate the latter just as accurately as before. The result must be, overall, a more precise estimate of the player's basic strategy advantage.

To do this we need new, revised, best estimates of deck favorability for the "unnatural" part of the blackjack game similar to those originally provided on page 25 for the entire game. These numbers would be:[E]

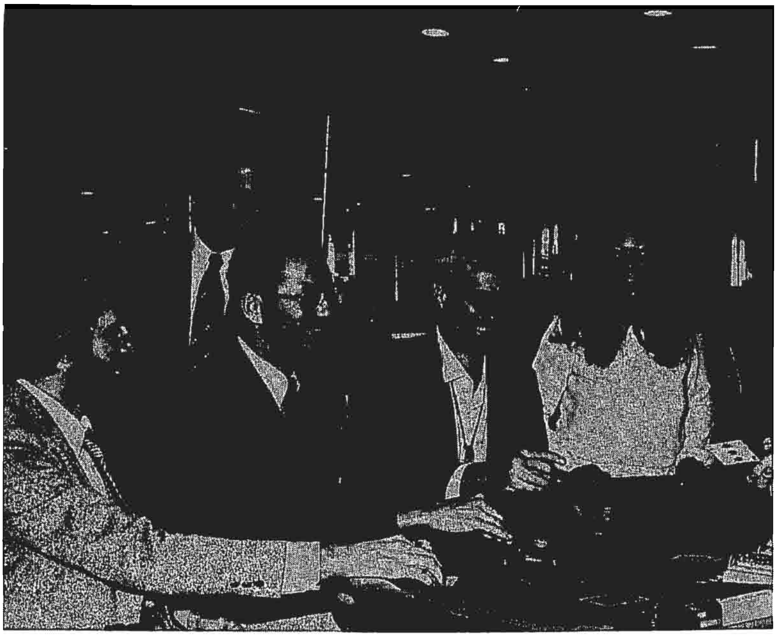
A	2	3	4	5	6	7	8	9	T
+3.8	−17.1	−20.1	−26.2	−33.4	−21.1	−12.0	2.3	11.4	20.6

Their average value for the full 52 card deck is −2.3(%) which is the player's expectation without the blackjack bonus. As a simple example of their employment, imagine a 26 card remainder of 10 tens, no aces, and two of each other denomination.

To estimate the non-blackjack part of the player expectation we add up the payoffs corresponding to the 26 remaining cards and then divide by 26:

$$\frac{10(20.6)+2(-17.1\ -20.1\cdots+11.4)}{26} = -1.0(\%)$$

Since there are no aces, a = 0 and our formula brings us the obvious bad news that we gain nothing from the blackjack bonus. Hence our final estimate of basic strategy expectation is $-1.0(\%)$ which differs from the $-.4(\%)$ which is obtainable by conventional methods.



From left, Stanford Wong, author of *Professional Blackjack* and *Winning Without Counting*; Ed Thorp, Author of *Beat the Dealer*, and Peter Griffin, author of *The Theory of Blackjack*, are photographed together for the first time, Oct. 24, 1981 at Barney's, at Stateline, Nevada. Looking on is Dr. William Eadington, Professor of Economics, University of Nevada, Reno, and a pit boss. Note Griffin preparing to catch a card in air.

APPENDIX TO CHAPTER 12

A.

In the subsequent discussion I shall assume that the player holds an abstract total (such as 14), is contemplating two actions (such as drawing or standing), and that the dealer possesses as an up card an abstract ace, thus leaving all 52 cards (including four aces) available for play. The assumption is, of course, artificial in that the player would have to have some cards and the dealer an ace for the strategic situation to arise. Nevertheless it serves two good purposes: to make discussion and reference simpler and to produce results compatible with existing and correctly computed tables of effects of removal.

Any poker player should know that there are $\binom{52}{5}$ = 2598960 possible five card hands and that all of these are equally likely if the deal is honest. The equiprobable assumption does not hold, however, in blackjack if the dealer has an ace (our "abstract" one) showing and has already checked to find a non-ten lurking as down card. The *a priori* and *a posteriori* distributions of five card subsets with respect to the number of tens present are as follows:

Number of Tens	Number of Subsets	A priori Probability	Probability of Non-Ten as Down Card	A posteriori Probability of Subset if Down Card is Non-Ten
0	376992	.146	1.0	.212
1	942480	.362	.8	.420
2	856800	.329	.6	.285
3	352800	.135	.4	.078
4	65520	.025	.2	.007
5	4368	.002	.0	.000

A formal treatment shows that the probability of a specified k card subset with exactly t tens in it becomes

$$\frac{13(k-t)}{9k\binom{52}{k}}, \quad \text{rather than} \quad \frac{1}{\binom{52}{k}}, \quad \text{all of this, of course,}$$

subject to the condition that the dealer's down card is a non-ten. If we let f(t) stand for the unconditional probability that a subset has t tens in it, then $\frac{f(t)(k-t)}{k} \cdot \frac{13}{9}$ becomes the revised, *a posteriori* probability of t tens in a k card subset which is playable (i.e. without a ten under dealer's ace). Hence the expected number of tens in a playable subset can be shown to be

$$\frac{13}{9k}\sum_{t} t\, f(t)(k-t) = \frac{16\,(k-1)}{51}, \quad \text{rather than } \frac{16k}{52}$$

Intuition provides the answer more quickly: one of the k remaining cards must be the non-ten under the dealer's ace; we expect 16/51 of the other (k−1) cards to be tens.

We can also derive the probability of removing a particular ten, such as the queen of spades, from a 52 card deck given the fact that the resulting 51 card subset is playable:

$$\frac{13\,(51-15)}{9{\cdot}51{\cdot}\binom{52}{51}} = \frac{1}{51}\cdot$$

Hence the probability that *some* ten was removed is 16/51 for 51 card subsets.

This last result explains why the effects of removal against a playable ace do not sum to zero: not all card removals are equally likely. In fact, assuming that the first 36 subscripted cards in the deck are the non-tens and that the 37th through 52nd are tens, we have

$$\frac{16}{51}\, E_{52} + \frac{35}{51}\, \frac{\sum_{j=1}^{36} E_j}{36} = 0$$

where Ej is the effect of removing the jth card. The equation follows from the fact that 51 card favorability is unaffected by all possible (and correctly probabilized) removals. Exact calculations, taking into account the player's revised chances of drawing either a ten or non-ten (see Epstein pages 224-5), of the effects verify this relation.

The violation of equiprobability of all possible subsets of the deck makes discovery of least square error estimates of strategic favorability a different and far more difficult task than that described on pages 32-35. However, an artificial expansion of the original sample space of all $\binom{52}{k}$ possible k card subsets will reestablish equal likelihood and allow employment of the Gauss-Markov equations.

We achieve this equiprobability by repetition of every k card subset with exactly t tens $13(k-t)$ times. The total number of points in 53-space to which we are trying to fit a least square hyperplane becomes $9k\binom{52}{k}$ rather than $\binom{52}{k}$. In this manner the probability of encountering a particular k card subset with t tens is $\dfrac{13(k-t)}{9k\binom{52}{k}}$, which we have already seen to be the correct distribution.

Although the general outline of the derivation follows the development in Appendix A to Chapter Three, it is far more complicated by the determination of the number of subsets in which individual cards and pairs of cards occur, as well as the total favorability of strategic action for subsets containing a particular card. It suffices here to present the solutions, β_j, which represent least square estimates of the favorability of strategy change in k card subsets for a situation where μ is the full deck advantage of the contemplated action and E_j is the effect of removing the jth card on that advantage. These solutions satisfy

$$k\beta_j = \mu - \frac{50}{(34k+16)}\left\{\frac{16(k-16)}{35}E_{52} + 35\,kE_j\right\} \quad \text{for } j = 1,36$$

$$\text{and} \qquad k\beta_j = \mu - \frac{50}{(k-1)}\left\{k + \frac{16}{35}\right\} E_{52} \qquad \text{for } j = 37,52$$

Note how the solutions are much more complicated than those for the conventional formulation with all subsets equally likely, namely $k\beta_j = \mu - 51E_j$. The right hand sides, or "single card payoffs" as they were called, are no longer independent of k. Also, the formula for β_{52} shows clearly how the ten-valued cards are granted ever increasing importance as the deck is depleted, and this reflects proper intuition.

Not only are the correct estimates of strategy change different, but also the variance of their distribution (necessary to produce tables such as GAINS FROM VARYING BASIC STRATEGY on page 30) will be different. The variance of the β_j derived here can be obtained, after excruciating calculations, as

$$\left\{\frac{576}{(k-1)}\left(13312+14720k+64244k^2-1241k^3\right)E_{52}^2 + \frac{(52-k)}{(34k+16)}\cdot\sum_{j=1}^{36}\left(16(k-16)\,E_{52} + 1225kE_j\right)^2\right\} \Big/ 44982k^2$$

This can be contrasted with the variance of the estimates erroneously based on the notion of equal likelihood, which is

$$\frac{51(52-k)}{52k}\sum_{j=1}^{52} E_j^2$$

The relation between this latter simple expression and the previous, accurate one depends upon k and the magnitudes of the E_j. For typical values of these quantities the actual variance tends to be slightly larger, but rarely by more than 5%. Hence there seems to be little reason to rework the existing approximations.

Indeed, the awkwardness of all the preceeding formulas suggests it would be a service to the users of the tables on pages 74 and 75 to artificially inflate the true effects of removal for a ten by multiplying them by 36/35. Then the sum of these altered effects will be zero and the conventional methods can be used as an approximation in order to avoid complication. This is what has been done on pages 74 and 75, so individuals wanting to use the methods of this Appendix will have to perform a preliminary multiplication of the tabulated effects of removing a ten by 35/36 to bring about the appropriate deflation and hence truly correct values.

If the dealer has a ten as up card similar arguments are applicable, only with the roles of ten and ace interchanged. It can be shown that best estimates of strategic favorability against a ten are given by analogous formulas:

$$k\beta_j = \mu - \frac{50}{(46k+4)} \left\{ \frac{4(k-4)E_1}{47} + 47kE_j \right\} \quad j = 5,52$$

and

$$k\beta'_j = \mu - \frac{50}{(k-1)} \left\{ k+4/47 \right\} E_1 \qquad j = 1,4$$

There are good reasons to avoid this complexity and to use the conventional methods in this case. Because there are few aces in the deck and because the strategic effect of an ace is generally quite a bit less than a ten (when the dealer shows an ace), the formally corrected estimates will rarely differ appreciably from those obtained assuming that the sum of the effects of all possible removals is zero.

B.

Although the figures in the table of critical ten count ratios are inferred from the formulas developed in Appendix A, exact calculations confirm the tabulated shifts in critical ten density for drawing and standing. It is no small source of wonder that the linear estimates suggest the player is .1% better off to draw than stand with 15 when there are six tens left out of 15 cards. The same estimate would be given for the specific "ideal" deck of six tens and one of each other denomination and would be wrong in that case by 1.6%, but is coincidentally right on the nose for the aggregate of all 15 card subsets containing six tens. Thus the estimation technique is not only unbiased for all possible 15 card subsets, but also approximately so for this special subclass which consists of those subsets with exactly six tens.

A comparison of the performance of the correct estimates of strategy obtained in Appendix A, the modified and simpler estimates resulting from conventional use of the page 74 tables, and actual calculations of optimal strategy is available from the approximately 250 non-insurance decisions which arose in the 5000 hand test described on page 61. The truly best estimates of Appendix A saved a theoretical 2.882 bets while the modified estimating technique made two unshared errors, lowering its gain above basic strategy to 2.678 bets. Absolutely optimal variation of strategy saved 2.969 extra hands when the dealer had an ace up, so the price paid by using the simpler estimates is a reduction in strategic efficiency (against the ace) from 97% to 90%.

The following case study exhibits 29 hands against a playable ace, 27 in which basic strategy should have been varied and two for which basic strategy was correct but linear estimation suggested a change. In the left margin of the table appears the player's total which is followed by the unplayed subset of cards which is coded as a ten-tuple with periodic spacing for legibility (thus 101 000 020 5 represents a nine card subset with one ace, one three, two eighths, and five tens). Next to the number of cards in the subset occurs the exact gain from drawing instead of standing or gain from doubling instead of drawing, the corresponding estimate from the truly best linear indicators of Appendix A, and finally the modified linear estimate just discussed, and labeled B. Comparison of the exact gains with the column A estimates lends insight into how well a computer programmed with linear strategy indices would do in actual play.

Total	Subset		Number of Cards	Exact Gain (%) From Draw/Dbl	Estimate A	Estimate B
18(Soft)	143 232 321	9	30	6.6	6.1	6.4
	131 333 122	5	24	1.0	.1	.4
17	120 413 122	5	21	.2	2.0	3.2
	221 411 313	3	21	24.0	26.1	27.3
	223 343 222	7	30	1.3	.5	1.2
	122 220 100	2	12	54.0	36.4	39.2
	011 323 213	4	20	−2.5	.7	1.8
	111 121 111	2	12	3.0	3.7	5.5
16	022 121 101	9	19	−4.6	.8	3.0
	223 101 423	7	25	−1.5	−1.7	−.9
	021 101 223	4	16	−1.8	−4.2	−2.8
	130 110 211	7	17	−12.8	−9.4	−7.2
15	221 210 112	5	17	−6.0	−5.6	−4.2
	142 231 234	11	33	−2.0	−1.4	−1.0
	323 340 443	12	38	−.1	−.9	−.6
	112 010 101	4	11	−13.0	−10.8	−7.3
	101 000 020	5	9	−67.0	−51.8	−45.2
	241 331 443	14	39	−3.2	−4.0	−3.9
13	111 000 020	5	10	−27.9	−9.0	−3.3
11	334 344 443	12	44	−10.2	−11.2	−11.7
	111 100 021	2	9	7.6	3.5	−1.9
	242 444 433	10	40	−18.5	−18.4	−19.1
	123 213 431	9	29	−2.1	−2.0	−3.2
	220 234 321	8	27	−7.1	−12.7	−14.3
	142 313 422	9	31	−1.2	−5.9	−7.0
10	022 122 101	9	20	14.3	9.0	6.0
	011 101 001	3	8	12.8	6.5	−4.1
	112 321 321	9	25	.7	−1.6	−3.2

C.

The method of analyzing continuous dealing, as well as analyzing the second hand itself, involves the definition of a discrete parameter Markov chain. For state space we select the possible subsets of cards from which a hand may be started. Fortunately there are only eight, which for convenience I will designate with letters. They are

A = {5TTT}	with expectation of	−1/5
B = {TTT}	" " " "	3/10
C = {TTTT}	" " " "	0
D = {55TTT}	" " " "	1/5
E = {555TT}	" " " "	−3/10
F = {55TT}	" " " "	2/5
G = {5}	" " " "	−3/28
H = {555TTTTTT}	" " " "	5/84

The one step (one hand) transition matrix may be obtained by tedious calculation and displays the probability of moving from one set of unplayed cards to another as a result of the player's use of basic strategy:

$$
\begin{array}{c}
\\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H
\end{array}
\;\frac{1}{420}\;
\begin{array}{c}
\begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\
\left(\begin{array}{cccccccc}
84 & 0 & 0 & 0 & 0 & 126 & 0 & 210 \\
84 & 0 & 0 & 0 & 210 & 126 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 420 \\
0 & 0 & 0 & 0 & 0 & 0 & 84 & 336 \\
0 & 126 & 0 & 0 & 0 & 0 & 0 & 294 \\
336 & 14 & 70 & 0 & 0 & 0 & 0 & 0 \\
180 & 30 & 60 & 150 & 0 & 0 & 0 & 0 \\
160 & 15 & 35 & 100 & 50 & 60 & 0 & 0
\end{array}\right)
\end{array}
$$

The long run distribution is described (approximately) by the vector (.311, .033, .052, .079, .053, .147, .016, .308), which in turn, when multiplied by the vector of expectations for each state, produces the long run expectation of 2.31%.

That both the second hand and long run expectations are lower than the full deck figure must be regarded as coincidental since other examples are possible where the change is in the other direction. For instance, had our original deck consisted of two fives and four tens, the first hand expectation would have been $2/15 = 13.3\%$, the second hand $31/150 = 20.7\%$, and the long run expectation precisely 20%.

It must be conceded that ordinary blackjack dealt continuously from a 52 card deck admits of the same type of decomposition. The number of states is, of course, intractably large and the long term effect is unlikely to be anywhere near as pronounced.

D.

The formulas to evaluate proportional wagering on favorable decks are derived by considering the player's advantage as a normally distributed variable with mean = m and variance = b^2. The player's wager will be proportional to

$$\int_0^\infty xN(m,b^2)$$

while his profit will be proportional to

$$\int_0^\infty x^2N(m,b^2)$$

The first integral is a linear loss integral, while the second one can be expressed as

$$\int_0^\infty x^2N(m,b^2) = m\int_0^\infty xN(m,b^2) + b^2\int_0^\infty N(m,b^2)$$

E.

To create these revised single card payoffs we need to know the effects of removal on the non-bonus aspects of blackjack. These are easily computed by subtracting the effects of removal on the blackjack bonus from the conventional effects shown on page 71. By creating the three generic 51 card decks we can compute −.49% as the effect on blackjack bonus for removing an ace, −.06% for removal of a ten, and .09% for removal of any other card.

The revised effects of removal for blackjack played without the natural bonus become

A	2	3	4	5	6	7	8	9	T
−.12	.29	.35	.47	.61	.37	.19	−.09	−.27	−.45

The payoffs, $\mu - 51E_j$, are computed with $\mu = -2.3(\%)$ to reflect the game's expectation without the blackjack bonus.

13

REGRESSION IMPLICATIONS FOR BLACKJACK AND BACCARAT

'Between the idea
And the reality
. . .
Falls the Shadow'
—T. S. Eliot

In statistics a *regression* function relates the average value of the predicted variable to the known value of some other, predicting, variable. As an example of this in blackjack, the predicting variable would be the known value of a count system (like a Hi Lo count of +3 with 39 cards left) and the predicted variable could be the expectation for the set of cards which gave rise to this count. It is common to assume that the predicted variable changes *linearly* with the predicting variable, that is, that there is a uniform rate of increase or decrease in the unknown variable regardless of the range in which the predicting variable lies. The blackjack analogy here would be the assumption that, for instance, the player's advantage changes at the rate of .5% per Hi Lo count per deck (and thus that the exampled $+3/39=+4/52$ count would presage a $4(.5\%) = 2\%$ rise in expectation.) In this chapter it will be shown why the actual regression function for best linear estimates of basic strategy expectation (an 'ultimate' count) is slightly nonlinear and what this portends.

The Problem

In Appendix A of Chapter Three it is shown that intuitive linear estimates of blackjack expectations, based upon known effects of removals of cards of different denominations, have the least squares property. For example, assuming one wants to estimate the basic strategist's expectation

Y_i as a linear function of the cards remaining in the i^{th} n card subset of a single deck, then

$$Y_i = \sum_{j=1}^{52} B_j X_{ij}$$

provides best linear estimation. The coefficient B_j equals $m-51E_j$, with m being the full deck basic strategy expectation for the rules considered and E_j the effect of removing the j^{th} card on that full deck expectation; X_{ij} is 0 or 1/n, reflecting the absence or presence in the i^{th} subset of card j. The estimator Y can be modified in fairly obvious ways for either multiple deck analysis or individual studies of strategy variation.

The effectiveness of these estimators for varying basic strategy has been studied extensively in Chapter 11. The method works remarkably well, resulting in a decision apparatus likely to accrue about 98% of the total expectation possible (gain above basic strategy) relative to computer-perfect playing decisions which would require calculation of exact probabilities to decide any situation.

There is sound intuitive reason to expect linear estimators to work well for individual strategy decisions: often only one unseen card is needed to resolve the situation, and very rarely are many (the most profitable strategy variation in single deck play, insurance, is completely linear in this sense). But when it comes to estimating the *basic strategist's* overall expectation, before the hand has been dealt, the fact that at least four cards (whose order is vital) will have to be used calls into questions the accuracy of this method.

An important and overlooked statistical fact is that the correlation coefficient between the least square estimates and the actual expectations being estimated is equal to the quotient of the standard deviation of these estimates and the standard deviation of the distribution of the actual expectations themselves. This leads us to the conclusion that *actual* expectations will have a greater dispersion than their surrogate best estimators. This is so because the correlation coefficient will be less than one for all but the N−1 card subsets of an N card deck. In particular, this underestimation of dispersion (using least squares estimates) will be most severe for the smaller subsets, in which linearity is likely to be poorest. Hence we want to learn more about how the number of unplayed cards is related to this correlation coefficient.

Because of the enormous amount of computer time necessary to evaluate basic strategy expectation for reasonably large subsets, no thorough investigation of this matter has been hitherto undertaken. Insight will be developed here from (a) a complete analysis of 'Woolworth Blackjack', played from a contrived deck of only fives and tens; (b) simulation and determination of exact basic strategy expectation for small subsets in the actual game of blackjack; (c) evaluation of the accuracy of Thorp's differential approximation to the infinite deck blackjack function; and (d) contrasting blackjack with baccarat, a simpler game which can be precisely analyzed much more quickly than blackjack.

Woolworth Blackjack

A Woolworth blackjack deck consists of only fives and tens; in particular let a single deck contain 20 fives and 32 tens. The player's optimal strategy for the full deck (basic strategy) is to double down with hard ten (two fives) against a five and stand with fifteen regardless of the dealer's card. The expectation relative to this strategy is −.63%.

There are only two denominations and thus only two effects of removal we are interested in, and they are easily calculated. Basic strategy applied to a 51 card deck with 19 fives yields an expectation of −.01%, while a 51 card deck with 31 tens has an expectation of −1.02%. Hence the effects of removal are +.62% and −.39% for fives and tens respectively.

The attractions of this simple game as an analogue for ordinary blackjack are immediately evident: (a) there are at most 21 distinguishably different n card subsets of a single deck and these can be completely probabilized in a trice; (b) best linear estimation is immediately inferred from either the density of fives or tens left in the deck; and (c) the variance of the distribution of best linear estimates is comparable in magnitude to that of ordinary blackjack, as is reflected by the similar removal effects.

There are two important quantities to record in an exhaustive analysis of all possible subsets of a Woolworth blackjack deck. The first is, of course, the correlation between the best linear estimates of expectation and the actual expectations themselves. This provides a commonly understood measurement of linear behavior and also indicates the ratio of the standard deviations of the distributions of the estimates and the actual expectations.[A]

The second quantity is what we shall term 'opportunity'. For any deck level (number of unplayed cards), define the opportunity to be the average amount of profit which can be gained by the player who wagers precisely one unit whenever the deck is favorable and nothing otherwise. Opportunity, then, is the sum of the products of positive advantages and their associated probabilities. A normal approximation to the distribution of linear estimates of advantage has been applied in Chapter 6 to estimate opportunity by use of the Unit Normal Linear Loss Integral. We will compare the actual opportunity encountered in Woolworth blackjack to its UNLLI estimate by presenting the latter in parentheses, following the exact measurement of opportunity.

The following chart presents correlation and opportunity figures for subsets of various sizes from both the single deck mentioned and an eight deck shoe (160 fives and 256 tens). The full eight deck expectation with the single deck strategy is −1.46%.

	Single Deck Subsets		Eight Deck Subsets	
Number of Cards	Correlation	Opportunity (%)	Correlation	Opportunity (%)
7	.67	3.60(3.19)	.67	3.45(3.07)
10	.77	2.77(2.51)	.75	2.67(2.42)
13	.82	2.21(2.13)	.80	2.16(2.03)
16	.86	1.83(1.76)	.83	1.82(1.78)
26	.93	1.07(1.09)	.89	1.21(1.24)
39	.98	.51(.53)	.93	.86(.91)
52			.95	.65(.69)
104			.98	.28(.31)
208			.99	.06(.07)

What is unexpected in the results is that the UNLLI estimate is much better than one would suppose, considering that it is based upon a distribution known to have smaller standard deviation than the distribution which is being approximated. A display of the entire regression function (with its linear approximation in parentheses) will shed light on the matter. We present such data for 13 and 39 card subsets of a single deck on the next page.

The most important characteristic to observe here is that subsets with either extremely positive or negative estimated expectations all have actual expectations well below those estimated values. Thus the subsets with more or less normal density of fives and tens, those most probable subsets in the middle of the distribution, have corresponding actual expectations mildly above the estimated ones. This provides the compensation necessary for the overall average of both the estimates and actual expectations to coincide at precisely the known full deck value of −.63%. The phenomenon is not hard to explain: although an extra ten in a nearly normal deck helps the player, a deck full of tens must produce only 20-20 pushes; similarly one extra five only mildly helps the dealer (who must hit 15), but when there are nothing but fives in the deck the basic strategist is forever losing suicidal double downs. We shall soon see

No. of Fives	13 Card Expectation (%)	Prob. of Occurrence	39 Card Expectation (%)	No. of Fives
0	.00(19.57)	.0006	−10.60(−6.38)	20
1	.00(15.53)	.0071	−7.91(−5.03)	19
2	2.56(11.49)	.0386	−5.55(−4.68)	18
3	4.54(7.45)	.1158	−3.52(−3.33)	17
4	4.61(3.41)	.2140	−1.78(−1.98)	16
5	2.33(−.63)	.2568	−.32(−0.63)	15
6	−2.45(−4.67)	.2054	.83(0.72)	14
7	−9.90(−8.71)	.1106	1.80(2.07)	13
8	−20.51(−12.75)	.0399	2.48(3.42)	12
9	−35.24(−16.79)	.0095	2.93(4.77)	11
10	−55.94(−20.83)	.0014	3.16(6.12)	10
11	−85.90(−24.88)	.0001	3.18(7.47)	9
12	−130.76(−28.92)	.000006	3.04(8.82)	8
13	−200.00(−32.96)	.0000001	2.73(10.17)	7

that this is also what characterizes the behavior of actual casino blackjack basic strategy expectation, as a function of best linear estimates.

The surprising accuracy of the UNLLI for estimating opportunity (despite too small a standard deviation) can be explained by the tendency to overestimate the advantages occurring at high counts being cancelled by the underestimation of the advantage for small counts near zero. The shape of the normal distribution is known to be a good fit for the distribution of probabilities for the *least square estimators*, but we now see that it is not such a good fit for the distribution of *actual advantages*, which distribution is quite skewed.

Digression: The Count of Zero

The question I have been most frequently asked in the past ten years goes, in its simplest form, something like this: 'Since the average distribution of 52 remaining cards with a count of zero from a four deck shoe is (almost) the same as a single 52 card deck, does it not stand to reason that the player's advantage in this situation is the same as it would be for the first hand from a single deck?' My response was to argue that since the average *basic strategy* advantage with 52 cards left in the deck must be the same as the full four deck advantage of −.49% and since the Hi Lo count is symmetrically distributed with 52 cards remaining (that is, a count of −1 is just as likely to occur as a count of +1, etc.) it made intuitive sense that the player's basic strategy advantage when the count was zero should be about the same as for the full four deck shoe.

I also wrote a computer program to test one particular aspect of the theory that a 0/52 count portended a single deck. Although it is hopeless to imagine calculating the player's expectation for all possible configurations of 52 cards with a zero count, there is one very important component of the player's expectation which can be calculated precisely, namely the blackjack bonus of an extra half unit paid to the player for a natural. If the advocates of the 'zero count equals single deck' theory were correct, then the blackjack bonus under these circumstances would be equal to, or at least closer to, its value of .023246 for a single deck than to the value .022718 which applies to the full four deck shoe. (See page 191.) The following table displays the worth of the blackjack bonus for a running count of zero with various remaining numbers of cards left from four decks.

Cards Left	**Chance of Uncontested Natural**	**Bonus Value**
52	.044637	.022318
104	.045216	.022608
156	.045406	.022703
208(full shoe)	.045437	.022718

So, not only was the blackjack bonus with 52 cards not worth the single deck amount, it was even worth somewhat less than in the full four deck shoe, hardly an augur of increased expectation! Also observe that the blackjack bonus's value of .022608 with 104 cards left is closer to the full four deck value than to the double deck value of .022892.

Another explanation for the presumed gain in advantage with counts of zero that gained currency is that it is due to variations in strategy that will take place on the hands dealt under such circumstances. But you can easily convince yourself that a gain in advantage due to strategy variation of .5% is just too much to occur for hands dealt with a zero count and 52 cards remaining. What possible changes in strategy can there be? If the dealer shows a ten the running count turns negative, and this rules out all but a very few stands on multiple card 16's whose combined worth would hardly total .02%. I can imagine no strategy changes against a dealer's 7 or 8, and only the most marginal and improbable gains again with hard 16 against a 9.

Insurance, the most productive strategy gain, is ruled out. All we're left with, it seems, is a few hard totals of nine to double (with marginal gain) against a 2 up (which would be a change from four deck basic strategy).

There may be a few other variations I've overlooked, but the total gain from card counting, using a point count system, would be no more than .04%. Even hypothesizing the most optimistic assumption of a full table with all cards exposed and taken into account, the gain would be no more than .30%. This is not to say that there isn't considerable profit to be reaped with strategy changes when 52 cards remain, but that profit will accrue from more extreme pre-deal counts, not zero pre-deal counts.

Having publicly, privately, and righteously condemned for many years this heresy that a count of zero indicates an increase in player advantage, I was stunned by the 2.33% expectation for a normal 13 card subset of 5 fives and 8 tens in Woolworth blackjack, this being almost 3% above the full deck figure. Suddenly I knew I had been very wrong, misled by my erroneous presumption that the distribution of player advantage was as symmetric as the distribution of a point count. What to do? Feverish recalculations only confirmed the error in my judgement. Could I conceal the finding, perhaps until after my death? Unlikely. I decided upon the route of confession: at least then *I* would be the one to prove myself wrong!

The correct explanation of this count of zero phenomenon goes like this: the player's *gain* in expectation for unusually high counts will be of a smaller magnitude than his *loss* in expectation for correspondingly negative counts. In the former situation, more pushes will begin to occur due to extra tens in the deck and double down opportunities will become less frequent. On the other hand, with outrageously negative counts, the basic strategist will often be doubling, splitting, and standing when the dealer is very unlikely to bust. For this imbalance to occur and yet result, as is provable, in no change in the overall basic strategy expectation, there must be a small rise in expectation somewhere in the middle of the distribution, quite possibly at counts close to and including zero.

To illustrate this, again using the Hi Lo count, one can calculate a basic strategy advantage of 18% for a +13 count with 13 cards remaining from a single deck. Note that this is below the 26% we would presume using .5% per true count. But for a −13 count with 13 cards remaining the basic strategy expectation is a whopping −135% because of the many hopeless doubles and splits. This is far below the estimated −26%. It therefore follows that for at least one of the running counts between −12 and +12 the actual expectation must be higher than the .5% per true count figure would indicate. This is because the overall expectation with 13 cards left must be precisely the full deck 0%.

To learn more about what happens in actual blackjack we must resort to simulations.

Actual Blackjack, 10, 13, and 16 card subsets

It turns out to be feasible to calculate exact basic strategy expectation for many subsets so long as they contain relatively few cards. (It takes 40 times as much computer time to analyze an ideal 26 card subset, two cards of each denomination, as it does to treat a 13 card remainder consisting of one card of each denomination.) To avoid running out of cards, we prohibit resplitting of pairs, but otherwise assume a total-dependent basic strategy for the Las Vegas strip game, for which the player's expectation is −.02% for a single deck and −.64% for eight decks (using single deck strategy). In the treatment of pair splits, only one of the two hands was evaluated and the resultant expectation was doubled. This surely introduces some distortion as there undoubtedly were some subsets, particularly ten card ones, for which resolution of all possible hands before running out of cards was not guaranteed. This bias does not appear important, however, since the average expectation of all 1000 ten card subsets was insignificantly different from the theoretical values in both one and eight decks.

	Single Deck Subsets		Eight Deck Subsets	
No. of Cards	Correlation	Opportunity (%)	Correlation	Opportunity (%)
10	.80	2.70(2.65)	.80	2.75(2.64)
13	.86	2.33(2.29)	.83	2.30(2.27)
16	.89	1.90(1.97)	.87	2.08(2.01)

(There were 1000 simulated subsets in each circumstance.)

Certain points are worth remarking upon: (a) the correlation between linear estimation and actual expectation is uniformly better for ordinary blackjack than for the Woolworth game. This suggests we can use the Woolworth figures at other deck levels as a lower bound for the actual blackjack correlation in the unsampled cases; (b) The UNLLI estimate of opportunity is very satisfactory; (c) the player is distinctly more likely to encounter a favorable set of cards than an unfavorable one. In the sampled region of the deck the player has a positive expectation roughly 60% of the time, averaging about +4%. This is balanced by a 40% chance of a disadvantage, averaging −6%.

Figure A contains a plot of 200 pieces of data from ten card subsets of a single deck and provides insight into the behavior of least square estimates. Note the bowed, parabolic, nature suggested for the regression function, similar to the shape we would observe if we plotted the 13 card regression function for the Woolworth game.[B]

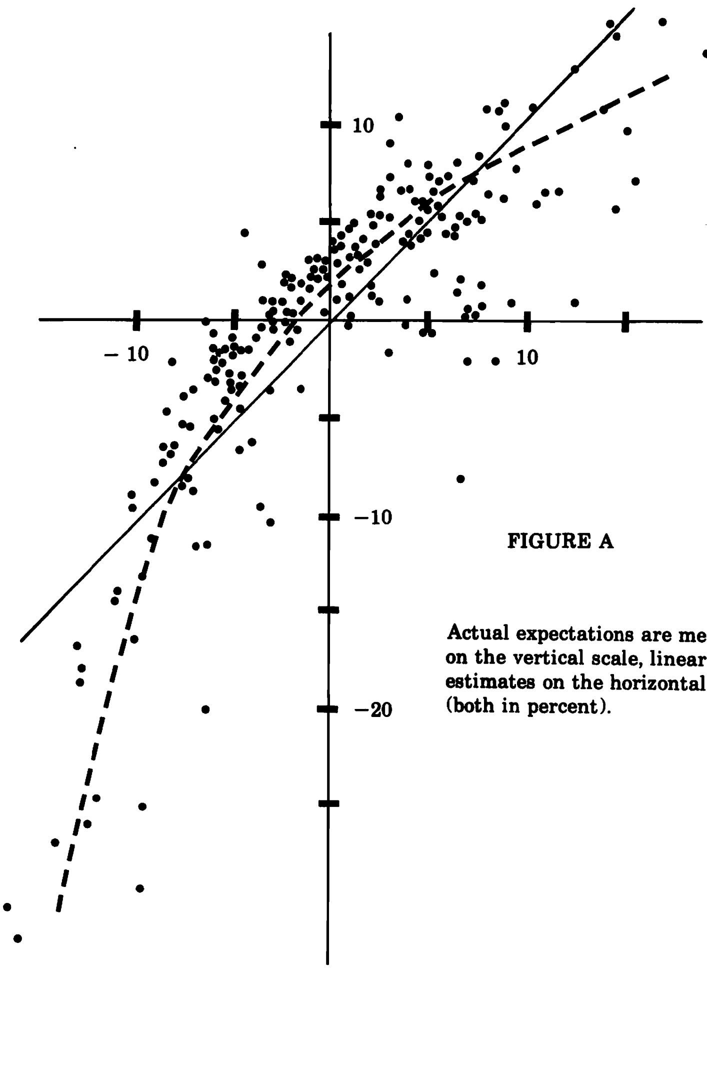


FIGURE A

Actual expectations are measured on the vertical scale, linear estimates on the horizontal scale (both in percent).

Other data gathered in these experiments shed light on the 'Count of Zero' phenomenon. The player's actual advantage, as a function of two well known card counting systems, the Ten Count and the Hi Lo, also displayed the same parabolic shape. Consequently counts near zero, reflecting normal proportions of unplayed cards, had actual expectations higher than linear theory would predict. For example, with 13 cards remaining, a Hi Lo count of zero was associated with an expectation of +1.60%, while 13 card subsets with precisely four tens remaining had a player advantage of +1.87%.

The most probable 13 card subset, one card of each denomination, had a 2.05% expectation for single deck basic strategy, 2.07% above the linear estimate of −.02%. The highest estimated expectation, 27.36%, occurs for 4 aces and 9 tens and is 3.44% above the actual expectation, while 4 each fours, fives, and sixes, and one three has the most negative estimated expectation of −28.39%, 138.03% above the actual value.

Linear Approximation to the Infinite Deck Blackjack Function

The phrase 'infinite deck blackjack' can be interpreted in two ways: either as the limit of considering increasingly large finite decks or as the result of dealing with replacement so the chance of a card's appearance at any stage of the hand is not altered by whether it has or has not appeared earlier in the deal. The two interpretations coincide, since the limiting probabilities in the first case are the same as those occurring from the independent dealing of the second.

Adopting the first perspective, we deduce that if all possible subsets of some fixed and finite size (52 for instance) were selected from an infinite deck (independent sampling), the average expectation of these subsets, probabilized *without* replacement, would be the same as for the infinite deck. We shall see shortly that this average expectation when probabilizing *with* replacement is not the same, but rather has a consistent and predictable bias from the full infinite deck figure.

Returning to the interpretation of independent probabilities, it is instructive to formulate the infinite deck blackjack function $E(p_1, p_2, \cdots, p_{10})$, the player's basic strategy expectation when p_i is the proportion, or probability, of card rank i in the mix. For our blackjack rules (no resplitting) the 'full' infinite deck expectation is E(1/13, 1/13, $\cdots$, 4/13) = −.690223%. It will be interesting now to pursue Thorp's (**Mathematics of Gambling**, 1984, Lyle Stuart, N.Y.) suggestion of a gradient vector for estimating the function E at other points, resulting in the following differential approximation:

$$E(p_1, p_2, \cdots, p_{10}) \sim -.69 - 52\sum_{i=1}^{10} p_i E_i \qquad (in\ \%)$$

where the E_i (−.59,.37,.43,.55,.69,.44,.26,.00,−.19,−.49 for i=1,10) are scaled to approximate effects of removal from a single deck.[C]

Intuitively we expect the approximation to be good near the central point (1/13,1/13,...,4/13) and poorer at points far removed. In order to test how well this linear approximation to E works, samples of 1000 subsets of various sizes were selected from an infinite deck (sampling with replacement). The correlation coefficients between the actual and estimated expectations were then calculated. Not surprisingly, the larger subsets, being closer on average to normal composition, had the higher correlations.

Number of Cards	Correlation Coefficient	Bias (%)
26	.93	−1.38
52	.96	−.69
104	.98	−.34
208	.99	−.17
416	.995	−.08

More interesting than the correlations, though, is the 'Bias' column which represents the discrepancy between the average of the actual and estimated expectations in the 1000 simulated subsets. It shows that using 'with replacement' calculations underestimates finite deck expectations by an average amount almost identical to the difference between the known expectation of a normal finite deck of the same size and an infinite deck. This suggests the following explanation of why basic strategy has a higher yield in smaller decks of normal composition: basic strategy intrinsically exploits the failure of small cards to reappear in the double down and standing options that only the player and not the dealer, can exercise.

Can Baccarat Be Beaten?

The one striking similarity between casino baccarat and blackjack is that both games are dealt from decks of cards which are not (usually) reshuffled after each hand. Hence the gambler is confronted with *dependent* trials for which the odds change from hand to hand. Consequently he might encounter occasionally favorable situations, as is known in blackjack, and exploit them by a dramatic increase in wagers.

A piece appearing in the March, 1982 issue of *Gambling Times* displayed certain six card subsets (the smallest number of cards for which the baccarat hand can be assuredly finished without reshuffling) which had some very positive expectations for bets on either 'Banker,' 'Player,' or 'Tie'. Joel Friedman's reaction to this article was to analyze *all* possible subsets of an eight deck baccarat shoe to determine precisely how often the bets would be favorable and how much could be gained by exploiting them.

At first glance Joel's task appears prohibitive because the total number of subsets is $\binom{416}{6} = 6{,}942{,}219{,}827{,}088$ and each six card subset would have to be looked at in $\binom{6}{2} \times \binom{4}{2} \times 2 = 180$ possible ways. Joel probably reduced the magnitude of his endeavor by observing that the number of distinguishably different subsets (taking into account the irrelevance of suits and distinction among tens, jacks, queens, and kings) was only 5005. At any rate, Joel communicated some of his results privately to me. I checked them with my own calculations, found them to be correct, and will now offer them to you so *I* will get credit for publishing first!

Six Card Baccarat Subsets

Wager	Chance it is favorable	Average Expectation when favorable(%)	Expectation per hand played(%)
'Player'	.150967	3.20	.4831
'Bank'	.270441	3.26	.8818
'Tie'	.339027	72.83	24.6909

The figures for the 'Tie' bet look quite promising until we read Ed Thorp's response which was published in the first edition of *The Experts* blackjack newsletter, in the summer of 1982. Thorp recounted how he and William Walden had first worked out the exact baccarat expectations in the early sixties (John Kemeny and Laurie Snell of Dartmouth College, are accorded credit for the first infinite deck, with replacement, approximate calculations; Thorp and Walden took into account the changes induced by the removal of each card) and also devised an effective card counting strategy to exploit the no longer existing 'Natural Eight' and 'Natural Nine' bets. Thorp further quoted from his 'Fundamental Theorem of Card Counting' paper and some limited simulations to conclude that, in his words, 'no practical card counting systems are possible.'

My understanding of current baccarat shuffling procedures is that six card subsets are a pipe dream, a fantasy never to be realized. In New Jersey, where the shuffling procedure is rigidly controlled, a 10 card subset *might* occasionally occur, but perhaps only once in fifteen shoes. In Nevada the possibilities are much more grim. At any rate, the following data illustrate how rapidly the spectrum of opportunity withers as the number of unplayed cards grows. For various numbers of remaining cards from an eight deck shoe, we present baccarat data similar to the previous blackjack data. The 'Player' and 'Bank' bets are combined since best linear estimates for one are almost equivalent to those for the other.

	Player & Bank Bets		Tie Bets	
Number of Cards	Correlation	Opportunity (%)	Correlation	Opportunity (%)
6	.30	1.36(.16)	.12	24.69(.33)
10	.64	.24(.07)	.35	2.98(.08)
13	.74	.12(.04)	.50	1.11(.04)
16	.78	.09(.02)	.52	.61(.02)
26	.89	.03(.004)	.73	.08(.003)
52	.94		.85	
104	.98		.96	
208	.99		.98	

(All possible six card subsets were analyzed. The rest of the data is from simulated subsets, 2000 with 10 and 13 cards, 1000 of size 16, 500 of size 26, and 200 of each of the others. With 52 cards left there was only one favorable bet, with expectation on 'Player' of .07%; there were no advantageous wagers found beyond this level.)[D]

Even 2000 simulated subsets are rather unreliable for the volatile 'Tie' bet, but the handwriting is definitely on the wall and it supports Thorp's contention. With 10 cards remaining (the most extremely optimistic assumption) we could expect to earn 3.22% of the wager we're willing to make on whatever favorable situations might occur then, however minuscule their advantages be. With 16 cards this drops to .70% and with 26 cards to about .11%. Beyond this level there's virtually nothing to bother with.

But, a typical shoe will rarely offer us wagers with precisely 10 cards remaining, even in Atlantic City. A smoothing and averaging of the simulated opportunity figures suggests that, if

(a) we had a computer capable of calculating exact expectations for encountered baccarat subsets and

(b) had enormous assets capable of funding $1000 bets whenever the shoe went good, to any degree, and on any bet,

then we would profit from our knowledge and technology at the rate of 2% of $1000, or $20 per baccarat shoe. A deeply dealt baccarat shoe takes perhaps an hour to deal, so it would seem there must be something better to do with our science! It is interesting to speculate that we would be unlikely to average more than *one* bet per shoe, and if we had to make 80 waiting bets at $25 each on perhaps a typical disadvantage of 1% (picking the best of the bad bets by our computer), our camouflage would eat up all our profit!

Ultimate Point Counts

Having whetted the reader's ravenous appetite for baccarat action, it wouldn't do for me to leave unfulfilled the promise that I would present in this book the most powerful card counting systems for diagnosing the favorability of the baccarat deck. Most of you, undoubtedly, do not have access to computers to guide your play, but casinos smile benignly on the practice of some players to keep a baccarat score card ('table de banque') for determining the next bet. Hence the complicated arithmetic necessary to use my ultimate point counts can be easily carried out with pencil and paper right at the table!

Note that in the very last row I extend the work of Thorp and Walden by presenting the player expectation for the various bets for the full eight deck shoe to one more digit than they published. Such are the miraculous advances in computing in the past 20 years.

Ultimate Point Count Values

Denomination	'Player Bet'	'Bank Bet'	'Tie Bet'
A	−1.86	1.82	5.37
2	−2.25	2.28	−9.93
3	−2.79	2.69	−8.88
4	−4.96	4.80	−12.13
5	3.49	−3.43	−10.97
6	4.69	−4.70	−48.12
7	3.39	−3.44	−45.29
8	2.21	−2.08	27.15
9	1.04	−.96	17.68
T,J,Q,K	−.74	.78	21.28
Full Shoe %	−1.23508	−1.05791	−14.3596

How do we use these numbers? Very much like any blackjack point count system. Suppose you want to monitor the 'Bank' bet. Begin the shoe with a running count of zero and then add the point values for all observed cards to maintain an up to date running count. Naturally, 'add' means 'subtract' if the value is negative. To estimate the instantaneous expectation of the 'Bank' bet at any time thereafter, merely divide your current running count by the number of unplayed (or unobserved) cards remaining in the shoe at that instant. Use the resultant quotient to adjust the full deck expectation of −1.057919%).

Example: Suppose the first hand out of the shoe uses a 3 and 4 for the Player and 9 and Jack for the Bank. Our running count is 2.69+4.80−.96+.78 = +7.31. Now, don't plunge into the 'Bank' bet just because you have a positive count! Rather, divide it by the number of remaining cards, which is 416−4=412. You estimate the 'Bank' expectation to be

$$-1.05791 + 7.31/412 = -1.04016\%,$$

so the shoe is not quite ready for us.

Similarly we would estimate the 'Player' bet at

$$-1.23508 + (-2.79-4.96+1.04-.74)/412 = -1.25316\%$$

and the 'Tie' as

$$-14.3596 + (-8.88-12.13+17.68+21.28)/412 = -14.3160\%$$

The actual figures for these bets with the 412 card subset are −1.04006, − 1.25326, and −14.3163 respectively and demonstrate the accuracy of the 'ultimate' counts for large subsets.

Before you start wondering why I'm offering these marvelous gambling aids to you at such a ridiculously low price (along with the ginzu knife and the wok) instead of trying to peddle them to some well healed sucker, I'll show you again how their accuracy diminishes with smaller subsets, precisely the ones we need to exploit if we're going to make any money at baccarat.

In another experiment, I had the computer select a single subset of various sizes and record the cards in these subsets as well as the associated player expectations. Here are the results, all expectations again in %.

	Number of Remaining Cards					
	312	208	104	52	26	13
Cards Left of each Denomination						
A	24	15	12	6	3	0
2	22	22	4	4	2	2
3	26	18	9	0	2	1
4	23	18	3	4	1	0
5	25	15	6	5	3	1
6	22	14	11	6	3	2
7	19	14	8	3	2	0
8	23	15	9	9	3	0
9	26	14	11	3	3	1
0	102	63	31	12	4	6
'Player'	−1.159	−.98	−1.48	−1.69	−1.94	−2.28
(estimate)	(−1.159)	(−.99)	(−1.59)	(−1.81)	(−1.91)	(−1.40)
'Bank'	−1.137	−1.30	−.82	−.61	−.36	−.09
(estimate)	(−1.137)	(−1.30)	(−.72)	(−.50)	(−.40)	(−.90)
'Tie'	−15.91	−14.3	−13.7	−12.2	−10.4	−33.0
(estimate)	(−15.83)	(−14.4)	(−14.8)	(−14.7)	(−11.2)	(−14.9)

(To test understanding of the use of these point counts, the reader should try to reproduce the figures labeled 'estimates.' Remember, the number of *removed* cards of each denomination is 32 minus the number remaining for non-tens and 128 minus the number remaining for tens.)

In none of these 18 sample estimations did our ultimate point counts mislead us into accepting an unfavorable wager (as all 18 bets were). The direction of the change in expectation (from normal, full shoe composition) was correctly identified for all 'Player' and 'Bank' bets, although not always for the 'Tie'. But what is most important to absorb from this study is that our error in estimation is growing appreciably as the shoe is being depleted and is greatest near the end, precisely when the favorable situations would arise, if ever.

The ultimate point count values provide us with further insight into the futility of counting down a baccarat shoe. For example, the cards whose removal most enhances the expectation of the 'Player' bet are fives, sixes, and sevens. Just imagine that, miraculously, the first 96 cards out of the shoe were the 32 fives, sixes, and sevens. Our point count would *still* not suggest an advantage for the 'Player!'

$$-1.23508 + 32(3.49+4.69+3.39)/320 = -.078(\%)$$

Actually the point count misses this situation, though, for the actual expectation with precise calculation is +.016(%), nevertheless hardly worth wagering on since the expected earning on a $1000 wager would be only 16¢! This also illustrates how the point count, although measuring the actual 1.251% *change* in advantage rather well as 1.157(%), is quite prone to being on the wrong side of the small advantage bets. And, of course, most of the advantages which occur in baccarat are extremely small.

More extensive simulations suggest the following about the ultimate counts' behavior:

(a) If you used them to pick the least negative expectation (rather than to raise your bet on putatively advantageous situations) and selected that wager suggested by the count values as best, you could improve on the −1.06% 'Bank' expectation (the best wager) by an average of .09% per hand. That is, betting the same amount on every hand, but picking the hand with highest estimated expectation, you would play with an expected loss of .97% of your constant wager.

(b) The ultimate count is worthless for diagnosing favorable 'Tie' bets. And of course, it is the 'Tie' which provides most of the opportunity to profit, small as it is.

(c) Betting on 'Bank' or 'Player' whenever the ultimate counts suggest an advantage, not wagering otherwise, would yield a profit of .07% of your agreed upon maximum bet per shoe in Atlantic City (virtually nothing in Las Vegas). Assuming you'll wager $1000 whenever you get the go-ahead, this translates into an expected earning of 70 cents per shoe = 70 cents per hour. In an eight hour day you might make three bets.

So, enjoy!

APPENDIX TO CHAPTER 13

A.

A Theoretical Question: Is the Correlation of Least Square Estimates a Monotonic Function of the Subset Size? It is relatively easy to show that the average squared error of estimation increases as the number of cards in the subsets diminishes. If X_{ij} and Y_{ij} are the actual and estimated expectations of the j^{th} n card subset of the i^{th} n+1 card subset, then their average values (summing on j) are X_i and Y_i, the actual and estimated expectations of this i^{th} n+1 card subset. Hence

$$\sum_{j=1}^{n+1} (X_{ij} - Y_{ij})^2 \geqslant (n+1)(X_i - Y_i)^2$$

Letting E_n be the average squared error with n cards left and summing on i over all n+1 card subsets produces

$$(N-n)\binom{N}{n}E_n \geqslant (n+1)\binom{N}{n+1}E_{n+1}$$

because each distinct X_{ij} occurs N−n times on the left hand side of the inequality. Our result, $E_n \geqslant E_{n+1}$, follows by cancellation.

Nevertheless, this falls short of establishing what seems intuitively evident, namely that correlation coefficients must increase with the subset size n. The difficulty is that the variance of the distribution of expectation also is larger for the smaller size subsets. (This is the major implication of Thorp and Walden's Fundamental Theorem of Card Counting.) Hence, since correlation is a measure of predictive error relative to variance, it might be the case that a correlation for n card subsets would be higher than that for the n+1 card subsets because the slightly greater error in prediction was swamped by the larger variance.

The structure of the variance in expectation (and consequently also that of the correlation coefficients) for any game dealt without replacement from a finite pack of N cards depends intimately on the minimal number of cards, J, necessary to *guarantee* resolution of the game. For baccarat J is equal to six, while for Woolworth Blackjack it is seven. Single deck blackjack, treated in this chapter, would require 20 cards left in the deck to assure resolution of the game without reshuffling, but the value of J would be much higher for multiple deck blackjack or if resplitting of pairs were permitted.

It turns out that specifying the variances (or correlations) for J distinct subset sizes determines the variance (correlation) for any other value of n. The following formula expresses the linear relation linking the reciprocals of the squares of the n card correlation coefficients, symbolized by R_n:

$$\sum_{i=0}^{J}(-1)^i \binom{J}{i}\binom{n+i-1}{J-1}(1/R_{n+i})^2 = 0 \qquad \text{for } J \leqslant n < N-J$$

(This formula was inferred from empirical data randomly generated by a computer. A proof, which is very lengthy and tedious, was later discovered. The general idea is to express all expectations in terms of the embedded J card subsets having $J-1$, $J-2$, $\cdots$, and $J-J=0$ cards in common, multiply by factors suggested in the formula, use a little theory of equations, and crown it all off with the intriguing combinatorial identity

$$\sum_{i=0}^{J}(-1)^i\, i^k \binom{J}{i} = 0 \qquad \text{for } 0 \leqslant k < J.)$$

A similar formula for the variances follows from the identity

$$(1/R_n)^2 = n\ V_n\,/(N-n)(N-1)\,V_{n-1}$$

where V_n is the variance in expectation for n card subsets.

Since it is always true that $R_{N-1} = 1$, there is also implicit a linear relation among J, rather than J+1, of the R_n. The coefficients also alternate in sign, but are more complicated than the above ones, depending upon N as well.

For example, when the game's expectation is determined by the J=2 card subsets and $n < N-2$, we deduce the relation

$$1/R_n^2 = (n(N-n-1)/R_{n+1}^2-(N-2))/(n-1)(N-n-2)$$

and in this case it is provable that $R_n \leqslant R_{n+1}$.

On the other hand, in the specific case J=3 and N=7 we obtain

$$1/R_3^2 = 9/R_4^2-18/R_5^2+10,$$

and, while we can prove that $R_4 \leqslant R_5$, there are admissible values of the coefficients of determination (1/2, 9/19, and 2/3 provide an example) with $R_3 > R_4$. Nevertheless, all efforts to contrive a game exhibiting correlations which violate the monotonic relations have resulted in failure.

B.

For these 200 points the correlation coefficients were .80, .87, and .89 for linear, quadratic, and cubic regression respectively. Beyond the third degree curve fit there was no significant decrease in overall predictive error. The characteristic of overestimating advantage at the positive and negative extremes and underestimating in the middle of the distribution undoubtedly occurs regardless of how many cards remain. As we move back in the deck (increase the number of unplayed cards) the correlation gravitates upwards toward one, both due to a flattening out of the regression function (into a straight line) and a migration of the experimental points closer to that regression curve (reduction of squared error from the regression function).

C.

Actually E is a polynomial function of nine, rather than ten, variables due to the constraint $p_1+p_2+\cdots+p_{10}=1$. The directional derivatives of E on the simplex $p_1+p_2+\cdots+p_{10}=1$ are easily estimated and essentially coincide with best linear estimates obtained from increasingly large finite decks. For example calculate E(399/5199, 400/5199, $\cdots$, 1600/5199) = −.696142%, which approximates the effect of removing an ace from 100 decks of cards as −.005919%.

D.

Observe how poorly correlated the linear estimates are, in contrast to Woolworth blackjack. Traditional card counting systems are futile because the only worthwhile wagers occur near the end of the shoe when correlations begin to disintegrate to the degree that the capacity to distinguish the favorable subsets is lost. Wagering on 6 card subsets diagnosed as favorable by linear estimation produces a combined profit of .20% per hand dealt (compared to the tabulated opportunity of 1.36%) for the 'Player' and 'Bank' bets, but a *loss* of .03% per hand on the 'Tie' bet. (The latter figure does show some discrimination, since indiscriminate wagering would lose at 14.36% per wager, whereas the linear estimates lost at only .37% per bet.)

Figures B and C present scatter diagrams of 100 data points each for the 'Player' and 'Tie' bets with 20 cards remaining in the shoe. They illustrate that the poor correlations (.83 and .77 respectively) are due more to the large deviations from the regression functions than to any peculiar non-linear nature of these curves.

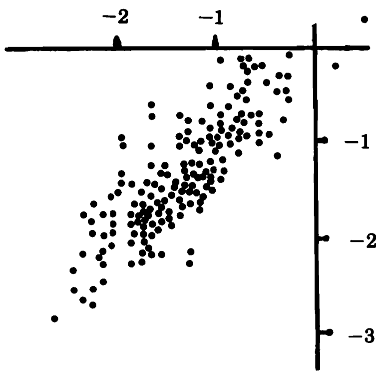


FIGURE B

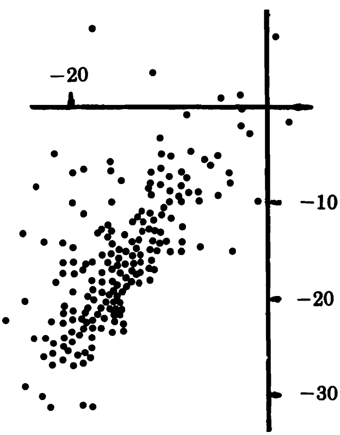


FIGURE C

Actual expectations are measured on the vertical scale, linear estimates on the horizontal scale (both in percent).

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POSTSCRIPT 1986

There once was a man from Japan
Whose limericks never would scan.
When people asked why,
He'd always reply:
'Because I invariably try to jam
as much into the last line as I possibly can.'

John Gwynn's 1984 epic simulation of the behavior of nearly optimal strategy devices in single deck blackjack showed that very time-consuming calculation of dealer and player probabilities did not appreciably improve the gain above that obtainable by using the lightning fast linear estimates from Chapter Six of this book [A]. Since single deck games are increasingly rare we present here, first, tables similar to those on page 28 and 30 showing both the strategy gain and its origin for four and six decks. Then we'll demonstrate how to modify the Chapter Six methods for multiple deck play.

	FOUR DECKS		**SIX DECKS**	
Number of Unseen Cards	Insurance Gain	Strategy Gain (No insurance)	Insurance Gain	Strategy Gain (No insurance)
52	.139	.91	.151	1.00
78	.082	.51	.097	.61
104	.048	.30	.065	.40
130	.025	.18	.044	.28
156	.009	.10	.029	.20
182	.001	.05	.017	.14
208			.009	.10
234			.004	.07
260			.001	.04
286				.02

No figures are presented for how much can be gained by betting an extra unit when the deck is favorable since tables on page 119 and 128 already cover this.

The next two charts can be compared with page 30 for single deck. They break down the sources of strategy gain in four and six decks into their origins, with tabulated figures being in 1/1000th of a percent. Both the four and six deck shoes were assumed to be dealt approximately four fifths through.

FOUR DECKS

Player				Dealer Up Card						
Hand	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**T**	**A**
17							2	1		2
16	1	1	1		1	8	6	8	55	1
15	3	2	2	1	1	3	2	2	18	1
14	6	4	3	2	2	1			13	1
13	15	11	8	4	6				7	
12	7	10	14	11	15				2	
11						1	1	2	5	5
10						1	1	3	5	2
9	4	4	2	1	1	3				
8			1	1	2					
Insurance										51
A9 and TT	1	2	4	6	7					
A8		1	1	2	2					
A7	2	1	1							
A6	1	1								
A5		1	1							
A4			1							
A3			1	1	1					
A2			1	2	2					

SIX DECKS

Player Hand	Dealer Up Card 2	3	4	5	6	7	8	9	T	A
17							1			2
16	1					6	5	6	47	1
15	2	1	1	1	1	2	1	1	13	1
14	4	3	2	1	2				9	
13	12	8	6	3	4				4	
12	5	8	12	9	12				1	
11							1	1	4	4
10						1	1	2	4	1
9	3	3	2	1		2				
8				1	2					
Insurance										38
A9 and TT	1	1	2	4	5					
A8			1	1	2					
A7	2	1	1							
A6	1	1								
A5		1	1							
A4			1							
A3			1	1	1					
A2			1	2	1					

That strategy gain diminishes as the number of decks in play increased was not unknown previously. Nevertheless, it is interesting to observe that the order of dominance for the two most important plays, 16 vs Ten and Insurance, reverses itself in the multiple deck game wherein 16 vs Ten becomes more important. An explanation of this phenomenon is that full pack unfavorability becomes more important than volatility in multiple deck blackjack where fluctuation in composition of the cards is much reduced.

Another of Gwynn's single deck findings worth remarking upon is that perfect surrender decisions are worth about .16% beyond basic strategy surrender when 75% of the deck is dealt. This is almost as much as can be gained by insurance. It should be noted that the surrender decision is not one well treated by the Chapter Six linear estimation technique. The reason is that one must compare three, rather than two, alternatives: hitting, standing, and surrender.

Multiple Deck Strategy Tables

The strategy tables which appear on pages 74–85 are designed to approximate optimal strategy decisions in single deck blackjack. A simple modification enables the user to employ them for multiple deck play. The discussion in the middle of page 72 and in Appendix B, page 95, indicates that what is necessary is a revised set of 11th column figures for full deck favorability in whatever number of decks might be in use.

To avoid different tables for different numbers of decks, the following tables present the appropriate 11th column figures for infinite deck play. Then the method of interpolation by reciprocals can be used to infer the appropriate 11th column figure for the number of decks assumed in play.

p74	**p75**	**p76**	**p77**	**p78**	**p79**	
A	**A(s17)**	**T**	**9**	**8**	**7**	
−7.93	−6.37	−16.47	−13.05	−12.40	−37.67	17
14.98	5.66	0.06	3.38	5.21	6.06	16
17.54	8.74	3.60	6.91	8.97	10.10	15
20.09	11.83	7.13	10.41	12.74	14.14	14
22.62	14.92	10.67	13.91	16.51	18.17	13
25.20	18.01	14.21	17.42	20.27	22.20	12
−3.39	0.07	6.02	6.95	12.07	17.07	11
−9.55	−6.92	−3.40	2.78	8.87	13.55	10
				−12.48	−6.76	9
					−26.99	8
0.73	6.77	3.45	8.24	−6.63		s 18 hit
					16.06	s 17 hit
				−13.59	−17.96	(A7)dble
					−6.76	(A6)dble

p80	p81	p82	p83	p84	p85	
6	6(s17)	5	4	3	2	
−52.05	−51.51	−47.80	−44.64	−41.45	−38.32	17
−27.72	−32.18	−28.23	−24.52	−21.15	−17.82	16
−21.21	−25.42	−21.83	−18.45	−15.40	−12.38	15
−14.70	−18.66	−15.42	−12.38	−9.65	−6.94	14
−8.19	−11.90	−9.01	−6.32	−3.90	−1.50	13
−1.68	−5.13	−2.61	−0.25	1.86	3.94	12
33.37	33.23	30.74	28.30	25.74	23.23	11
28.78	28.25	25.63	23.05	20.32	17.64	10
12.10	11.76	8.50	5.30	1.96	−1.33	9
−2.79	−2.80	−6.73	−10.52	−14.42	−18.27	8
−16.75	−14.89	−19.55	−23.93	−28.32		7
−12.84	−11.33	−15.78	−20.01	−24.10	−28.10	(A9)
−1.64	0.79	−3.35	−7.20	−10.85	−14.44	(A8)
9.81	13.16	9.57	6.11	2.93	−0.20	(A7)
12.81	12.45	9.12	5.93	2.61	−0.66	(A6)
8.09	8.80	5.25	1.85	−1.63	−5.05	(A5)
6.15	7.03	3.40	−0.09	−3.64	−7.14	(A4)
4.06	5.14	1.41	−2.17	−5.80	−9.40	(A3)
1.81	3.09	−0.74	−4.41	−8.13	−11.82	(A2)

Here's how to modify the procedure explained on pages 72 and 86. Suppose you are playing blackjack from k decks.

I Infer the proper 11th column figure for k decks by the formula $m_k = (m_1 + (k-1)m_\infty)/k$, where m_1 is the 11th column figure for single deck (pages 74–85) and m_∞ is the 11th column figure for infinite deck play (presented here).

II Sum up the effects of removal (pages 74–85) for the n observed cards, multiply this total by 51, and divide the result by (52k-n).

III Adjust the result of I by adding the result of II.

As an example of the sensitivity of this multiple deck linear estimation we will practice the previous procedure to confirm that one should hit (T,2) vs 4 in k=7 decks, but stand in k=8 decks. From page 83 we obtain $E_2 = -.23$, $E_4 = -1{,}21$, $E_T = 2.50$, and $m_1 = .65$, while $m_\infty = -.25$ from the table in this section. Since three cards are seen, n=3.

When k = 7 our calculations are:

I $m_7 = (.65 + (7-1)(-.25))/7 = -.12$

II $51(2.50-.23-1.21)/(364-3) = +.15$

III $-.12 + .15 = +.03$ (%). Therefore draw a card.

For k = 8,

I $m_8 = (.65 + (8-1)(-.25))/8 = -.14$

II $51(2.50-.23-1.21)/(416-3) = +.12$

III $-.14 + .12 = -.02$(%). Therefore stand.

Unbalanced Point Counts and the Pivot

Earlier in the book I expressed the view that it is desirable for a count system to be 'balanced' in that the sum, and hence average, of the point values be zero. Many years ago Jacques Noir, in **Casino Holiday**, proposed assigning the value −2 to all the tens in the deck and +1 to the non-tens. Thus, in a single deck, the sum of point values was +4. Whenever the running count for this system reaches +4 the player has the assurance that there are precisely twice as many non-tens as tens remaining in the pack. Hence Noir's count provides an infallible and simple rule for perfect insurance bets, using running count alone.

Although several entrepreneurs copied Noir's count system, the understanding and exploitation of unbalanced counts lay dormant until 1983 when Arnold Snyder published **Blackbelt in Blackjack**, a helpful book for beginners. In it Snyder presented his 'Red-Seven' count, which was identical to the Hi Lo system except that the red sevens in the pack are also treated as small cards and assigned the value +1. Snyder introduced the term 'pivot', the sum of the point values for the whole deck, and made use of the fact that the pivot provides a single fixed point of reference for the deck's *average* condition which can be located by running count alone. A mathematical justification appears in Appendix B.

The Reverend Snyder, in advocating his Red-Seven count, appears not to heed **Proverbs 11:1** wherein we read

> *'A false balance is abomination to the Lord,*
> *A just weight is his delight'.*

It should be pointed out that a balanced count system also has a pivot, namely zero. My preference for balanced counts is not just based on the fact that they are easier to analyze: a pivot of zero locates (for running count players, trained in basic strategy) a more useful and common point of reference, namely normal full deck composition.

Volatility of Advantage for Various Rules

On pages 126—130 the volatility of Double Exposure and Early Surrender blackjack is described. The large magnitudes of the effects of removal of different cards with these rules resulted in much faster changes in advantage than in ordinary blackjack. A natural consequence was that casinos took bad beatings in each of these games: early surrender is not likely to reappear except by occasional and temporary mistake, while the rules for double exposure have been greatly restricted so that there is little current interest in the game.

Two other variations in the rules which do remain common are worth discussing in the context of their impact on the volatility of player advantage. Compare the following figures with those on page 44.

Rule	A	2	3	4	5	6	7	8	9	T	Sum of Squares
Ordinary Surrender	−.61	.41	.50	.64	.84	.49	.25	−.03	−.23	−.57	3.56
Dealer hits soft 17	−.55	.39	.46	.59	.70	.48	.27	−.00	−.21	−.54	3.01

If doubling down after pair splitting is permitted, the major changes are greater effects for removing a 4 or 5 (by +.02) and a ten (by −.01). Thus this rule also results in slightly greater volatility. The effects of removal on the blackjack bonus quoted in Appendix E, page 202, would also apply in the 2 to 1 blackjack payoff giveaway at Binion's Horseshoe in Las Vegas the week before Christmas: to the effects above, for dealer hitting soft 17, add −.49 for an ace, −.06 for a ten, and +.09 for all the others.

Some Very Important Information

Pages 173—178 present the precisely correct basic strategy for any number of decks and any rules, at least for all two card player hands. Refraining from doubling (A,4) against a 4 when the dealer stands on soft 17 in 27 or more decks is the final two card change of strategy as the number of decks increases. However, if we consider multiple card hands we must go to 125 decks before the basic strategy stabilizes: stand on (5,5,5,A) vs Ten in 124 decks, but hit in 125 decks.

On the other hand, if one defines basic strategy in terms of what to do after pair splitting, then there is no maximal number of decks beyond which basic strategy doesn't change. With unlimited pair splitting there would always be the possibility of splitting enough eights against a ten, getting (8,4,4,5) on the first several splits, and then correctly standing with (8,4,4). For example, with only one split permitted, it would be appropriate to stand on (8,4,4) after developing (8,4,4,5) on the first hand if 150 decks were in play, but the (8,4,4) should be hit with 151 decks.

Kelly Criterion Insurance

The following story is a real rumor. A popular pundit and apostle of Kelly criterion betting (whom we'll call 'Jay') was in Atlantic City on September 15th, the day the counters were allowed back in the casinos. J was particularly interested in checking out the Chocolate Nougat Casino's claim that 'they'd deal all but one out of 52' and allow any bet spread. J sat down at the quarter table and played a few hands at 25¢ a pop until he glanced at the five foot long shoe from which the cards were dealt. They were dealing 52 *decks* with the cut card one *deck* from the end!

Naturally irritated by the deception, J nevertheless decided to stick it out in hopes that the shoe would 'go good' since his virtually infinite bankroll could tolerate the attrition of the quarter, waiting, bets. Sure enough, J's patience paid off. Seven hours later, nearing the end of the first shoe, J realized the last 103 cards consisted of precisely 70 aces, 33 tens, and nothing else.

With lightning calculation he deduced an advantage on the next hand of 41% and an average squared result of 1.77. His reaction was immediate 'Marker, $231,638.42.' The pit supervisor rushed over with pen and marker pad and counted out the chips. J stacked it all in the betting square; he was betting his *optimal Kelly fraction* 41%/1.77 times his then current bankroll of a million dollars!

There was a hushed silence as the cards were dealt: ace for J, ace for the dealer, and then a ten on J's ace – a blackjack on his gonzo bet! 'Insurance?' the dealer asked.

J paused. He knew there were 32 tens in the remaining 100 cards, less than one third of those left. Slowly J said 'Mark $91,186.44 worth of insurance.'

The pit boss intervened. 'I'm sorry, sir, you'll either have to insure the whole thing or not take any insurance at all. House policy on bets this size.' 'Very well,' J said, 'pay me even money.'

What was J doing, making a negative expectation insurance bet and what was the pit's reaction?

Well, J's avowed principle in gambling is to maximize the average logarithm of his capital and not necessarily his expected capital itself and he had finally encountered a situation where pursuing the Kelly criterion required the acceptance of an unfavorable bet. With 23% of his capital bet and a 32% chance for the dealer to have blackjack, J actually only wanted to insure about 79% of his bet and that is what he attempted to do. However, when the choice was limited to insuring all or nothing, he chose what for him was the better course since the average logarithm for his rate of capital growth was $\log(1.2316) = .2083$ with the insurance as opposed to $.68\log(1.3475) + .32\log(1) = .2028$ without it.

What was the pit's reaction? Unknown to J, his deliberate style of play had long ago caught the attention of counter catcher Abram Carter who had been using the Roberts' ten count to case the shoe from the catwalk. Carter, assigning -2 to each ten and $+1$ to all the non-tens, had a running count of 204 and, although he was oblivious to the ace-richness of the 100 remaining cards, he knew that insurance was not warranted until the running count exceeded 208 in a 52 deck game. Consequently he signaled the pit that J had taken a sucker insurance bet and that they should comp the rest of his stay in hopes that they could get their money back. As a result, Jay's logarithm grew unboundedly ever after.

Since it may come as a surprise to many who believe in optimal proportional betting that they should occasionally take negative expectation insurance bets, a few guidelines are in order. A Kelly bettor should consider insuring at least a portion of his blackjack against a dealer's ace if p, the proportion of unplayed tens in the deck, exceeds 1/3(1+f), where f is the fraction of capital the player has bet. Note that this fraction is somewhat less than 1/3, which is the critical fraction for card counters trying to maximize their expected wealth rather than, as the Kelly criterion decrees, optimizing the average logarithm of their wealth.

The correct proportion of the blackjack to insure for these Kelly bettors is $x = 3p + \frac{3p-1}{f}$, where p and f are as described in the previous paragraph. As an example, suppose a player had bet $100, which was f = .05 = 5% of his then current bankroll of $2,000. After he turns over his blackjack and sees the dealer's ace he might realize there were 8 tens left out of 25 cards, so p = 8/25 = .32. The formula gives $x = 3(.32) + \frac{.96-1}{.05} = .96 - \frac{.04}{.05} = .96 - .80 = .16.$ so he should insure 16% of his bet, that is buy $8 worth of insurance.

If it's a choice between insuring all or nothing, as Jay faced, insurance should be taken if p is greater than $1 - \frac{log(1+f)}{log(1+3f/2)}$

We can even imagine an intemperate gambler who foolishly bets all of his fortune (f=1) on a hand of blackjack. Then, so frightened is he by the realization of what he has done, he repents and is converted to Kelly proportional betting just before he picks up his cards: 'Lord, I won't do it again, I'll always try to maximize my expected logarithm from now on, if you'll only let me win this one hand.' Assuming that the god of twenty-one is impressed by this sniveling appeal, he'll give the wretched gambler a blackjack. But then the devil steps in to test our man's new found faith – he gives the dealer an ace as up card!

Plugging $f=1$ into our first formula we see that the contemplation of insuring at least a portion of the bet should begin with values of p just above $1/3(1 + 1) = 1/6$, or when just more than one out of six remaining cards is a ten. If it's a choice between all or nothing, the critical fraction of tens is, from our last formula, $1-\frac{log(1+1)}{log(1+3/2)} = .2435$ or just less than one ten out of four cards.

Unfortunately our newly converted gambler has no money left to insure with. Will the casino accept his blackjack as security and mark it? Is the pit boss the devil?

The Small Player

Can it be that all important mathematical questions relating to gambling have finally been answered? Or is the reason for this final section the more likely explanation that I've written myself dry, that I'm out of original ideas, and that I've been forced to turn for inspiration to a casino coupon book which fell into my hands during my last visit to Las Vegas?

Desperate for material as I nursed my cool refreshing glass of white wine at the Frontier Hotel's Bar None Bar and listened to the better of their two Country and Western bands, it struck me that I might just be able to fill up a few more pages by evaluating the worth of the booklet from which I'd just torn the second 25¢ drink coupon. The more I thought about it, thumbing through Frontier Fling book #98623A, the more it seemed to me the public wanted and needed, nay, deserved sound advice on how to extract the maximal expected value from coupons. And so in what follows you will find my analysis leading to the conclusion that the booklet had a monetary value of $3.25 as well as some hints for achieving this expectation at minimal risk.

1. The first coupon in the book is the best. It's an offer of six one dollar gaming tokens for a five dollar buy in. Optimal play is to buy the six chips and trundle them immediately to the cashier, where you will receive six dol-

lars. One dollar gained at absolutely no risk! Resist all temptation to gamble with these chips.

2. The second coupon is a wooden token for a free pull on a giant slot machine. I've never heard of anybody winning anything with one of these, but if the line isn't too long you might want to indulge your 'something for nothing' streak and get a little exercise at the same time. Besides, I have it on good authority that all electricity in Nevada is generated by small dynamos attached to the slot machine handles, so you'll be doing a public service.

3. The third coupon offers $2.00 worth of nickels for $1.50 in cash. Need I say more? A riskless gain of 50 cents. But don't, under any circumstances, follow any suggestion to put these in one of the machines. Not only do you destroy your mathematical advantage, but it might take you as long as two hours to get them all to stay in the machine since occasionally a few squirt right back out.

4. The two 'Introductory Keno Tickets' can be disregarded immediately since they require you to put up 50 cents and the payoffs are exactly half of those for the normal dollar Keno ticket. Totally free Keno plays are, of course, worth something and should be exercised. I myself won $5.00 with one at the Ponderosa Hotel in Reno while 'doing' a coupon book I found at a gas station.

5. The next coupon is a '3 for 2' coupon which must be played at the blackjack table. Since the Frontier deals a six deck game, the basic strategy player has an expectation of about −.6% per resolved bet and believe me we intend to wait for a decision on this one. It may seem surprising but the blackjack player only wins about 47.5% of the resolved hands (he gets close to even by the bonuses and double downs), so I evaluate this coupon as being worth 46 cents. Here's how I arrive at it: we expect to lose .6% of our $2.00 bet, or 1.2 cents, while our coupon itself brings in 47.5 cents per play, being 47.5% of the extra dollar they pay when we win.

Card counters may want to count down the shoe for a few minutes until they spot a 'rich' deck before plunging in with the $2 bet, but I advise against this. They will almost certainly be dealt a blackjack in such a situation and then experience a great disappointment when they discover they are not paid the 3 to 2 bonus on the coupon.

This brings up the question of whether you should double down or split with another coupon if you have one. The point is moot at the Frontier since they won't allow you to, but there are other casinos which may permit it. In such cases you should always match a double down with a coupon since double downs are more likely to win than lose and hence you're bettering your normal 47.5% chance of winning on the coupon. Also match coupons on those splits for which you have a positve expectation, such as nines or eights against a small card or aces against anything. Other splits might also present some marginal advantage for matching coupons, but it's not worth getting into the subject since they're rare.

6. There is also a '7 for 5' coupon to be played at blackjack and its value is 92 cents. As before we win with the coupon 47.5% of the time, and this chance times our $2.00 bonus makes 95 cents to the good. But don't forget to subtract the $5×(−.6%) = 3 cents we expect to lose on our $5 bet.

7. The '3 for 2' Roulette coupon is worth 37 cents and is most ideal for team play. Bet color, even, or odd and you will win $3.00 18 times out of 38 and lose $2.00 the other 20. Hence you should profit 3×18−2×20 = 54−40 = 14 dollars for every 38 coupons, or 37 cents per coupon.

If you are with a gambling associate and both of you possess these coupons it is worthwhile to consider playing them at the same time, one of you betting red and the other black. By doing this you can't change the 37 cent value of each coupon, but you can reduce the long term fluctuation in your gambling capital.

To see this mathematically we must calculate the variance of your results in both possible ways you two can bet the coupons, since it is variance which determines the relative riskiness of two different gambles with positive expectation. Variance can be calculated as the average squared result minus the square of the average result.

For a single '3 for 2' coupon play the player wins three units 18/38 of the time and loses two units 20/38 of the time for an average squared result of $EX^2 = 3^2 \times 18/38 + 2^2 \times 20/38 = 242/38$. Hence his variance is $EX^2 - (EX)^2 = 242/38 - (14/38)^2 = 6.233$ squared units. For two partners playing independently at separate times or tables the variance of their combined gamble will be the sum of these two identical variances, or 12.47 squared units.

Now observe how the variance, and hence the risk, is diminished if the partners play on the same spin, seeming to bet against each other. In this case they will win one unit 36 times out of 38 and lose four units two times, namely when zero or double zero appears. The expected squared result becomes $1^2 \times 36/38 + 4^2 \times 2/38 = 68/38$ and the variance for their simultaneous play becomes $EX^2 - (EX)^2 = 68/38 - (28/38)^2 = 1.25$ squared units, quite a reduction from the previous 12.47!

Playing your blackjack coupons on other people's hands (assuming they will play the basic strategy, at least) and using them on double downs and appropriate splits is another way to maintain expected value while lowering risk.

So now you see how I came up with a \$3.25 value for the gaming coupons:

(1)	Free dollar token	\$1.00
(2)	Questionable free pull	—
(3)	Free nickels	.50
(4)	Keno-don't play	—
(5)	3 for 2 Blackjack	.46
(6)	7 for 5 Blackjack	.92
(7)	3 for 2 Roulette	.37
		\$3.25

Whether there is any value in the two 25 cent drink coupons and the 50 cent Chuckwagon buffet discount depends of course on whether you would freely indulge in these activities were it not for the inducement of the price reduction. But as for me, I had no choice but to use my two drink coupons since I was unable to time my plays at the roulette and blackjack tables with the arrival of the cocktail waitress.[C]

APPENDIX TO CHAPTER 14

A.

What follows here is a prescription for analyzing a multiparameter card counting system without using multiple correlation coefficients. The method was used to enable Gwynn to measure the ultimate capability of both the Gordon and Einstein counts with side counts of five other denominations (discussed in Chapter Five) in his historic 1984 simulation.

The problem is to create a single parameter 'effects of removal' count that exploits exactly the same information as a level one system (card values +1 and −1) supported by a side count of precisely J of the other zero-valued denominations. Here, in outline form, is how to do it; a subsequent example will illustrate the procedure.

I Assign the correct effects themselves to the side counted cards.

II (a) Assign, temporarily, the average of all low card (+1 value) effects to each low card.

(b) Assign, temporarily, the average of all high card (−1 value) effects to each high card. This will not be necessary if the tens are the only high cards, as with Gordon and Einstein.

III (a) If all of the zero-valued cards are side counted, you are finished.

(b) Otherwise assign to the uncounted zero-valued cards the sum of the J side counted cards' effects divided by J−13 (the division accomplishes an averaging and a change of sign).

(c) Add to each of the previously determined effects in II the quotient of the difference in the sums of the original correct effects and new effects determined in III (b) of the uncounted zero-valued cards by the number of denominations in the primary count.

The result will be that the newly determined effects will sum to zero and produce a single parameter system which has precisely the same correlation with the original effects as the appropriately determined multiple correlation coefficient of Chapter Five. As an example we will use the Einstein count with a separate side count of both sevens and eights and consider playing a total of 13 vs Ten, the effects for which are found on page 76:

A	2	3	4	5	6	7	8	9	T
.00	.45	.40	.20	−.26	−.43	−3.22	−3.48	.88	1.36

The outlined steps proceed:

I $E_7 = -3.22$ and $E_8 = -3.48$

II (a) $E_3 = E_4 = E_5 = E_6 = (.40+.20-.26-.43)/4 = -.02$.
(b) Unnecessary: $E_T = 1.36$.

III (a) $J = 2$, so continue.
(b) $E_A = E_2 = E_9 = (-3.22-3.48)/(2-13) = .61$.
(c) Add to all effects in II the quantity $(.00+.45+.88-.61-.61-.61)/8 = -.06$, giving

A	2	3	4	5	6	7	8	9	T
.61	.61	−.08	−.08	−.08	−.08	−3.22	−3.48	.61	1.30

If one did not distinguish the sevens from the eights, but instead counted them together as a block of cards of equal value, then step I would be modified to assign the average effect of the blocked, side-counted cards. In the example we would have $E_7 = E_8 = -3.35$, but the remaining steps would be the same.

An important thing to emphasize for those who employ multiparameter adjustments in actual play is that there can be new strategy decisions which have no conventional index for the primary count alone. The previous example of 13 vs Ten with the Einstein count illustrates this. A glance at the appropriate row 13 versus Ten on page 102 shows that there is no value of the Einstein (or Hi Opt I) count for which standing is preferable to hitting.

What must be done to implement the added information provided by, for example, a side count of the block (7,8) is to create an artificial, linearly based, index for the primary count. Caution, however; this artificial index should only be used with a highly correlated side count, not with the primary count alone. To find this index, begin by determining the average effect of a card in the primary count as

$$(4(1.36)-.40-.20+.26+.43)/8=.69$$

in the fashion indicated on pages 56 and 58. Then divide the 11^{th} column full deck favorability for hitting 13 vs T of m = 10.13 by this average effect of .69 to obtain the critical 'true count' standing value as 10.13/.69 = +15 points per full deck.

An explanation of how much to adjust the running count for each extra or deficient side counted card already appears in Appendix C to Chapter Five on pages 62–64, provided there is only one block being tracked. The following example illustrates what to do if another block of cards is also monitored. Suppose, again, 13 vs Ten with a primary Einstein count and a side count of two blocks, (7,8) and (9). To determine the appropriate running count adjustment for each of these blocks do the following:

I Divide the sum of the effects of all J side counted denominations by 13–J: (−3.22−3.48+.88)/(13− 3) = −.58.

II Add the figure computed in I to the average effect for each block
(a) −3.35−.58 = −3.93 for (7,8).
(b) .88−.58 = .30 for (9).

III Divide the figures in II by the average effect in the primary count to obtain the running count adjustment
(a) −3.93/.69 = −5.7 points for (7,8).
(b) .29/.69 = .4 points for (9).

When the adjustment is small, as in III (b), it is probably best to ignore this denomination in actual play.

B.

Let there be N cards in the whole pack (for single deck N=52, etc.) and define the 'pivot', P, as the sum of the point values for the full deck. When n cards remain, the average, or expected, running count will be (N–n)P/N, and the distribution of running counts will have a standard deviation of $b\sqrt{(N-n)n}$ where the constant of proportionality b is importantly free of n.

Suppose A is any other variable (such as Advantage) in the blackjack game that one is trying to estimate using the count. Then the standard deviation of A will be $a\sqrt{(N-n)/n}$ where, again, a does not involve n. Using the assumption of a linear conditional mean, the average displacement (or change) in A given the current value of the count will equal the product of the correlation, ρ, between A and the count, the standard deviation of A, and the standardized value of the count. When the running count equals the pivot, this becomes

$$\rho a \sqrt{\frac{N-n}{n}} \cdot \frac{P-(N-n)P/N}{b\sqrt{(N-n)n}} = \frac{\rho a}{b} \cdot \frac{P}{N},$$

which is independent of n, the number of cards left. In fact, only when the running count is equal to the pivot will this phenomenon take place.

C.

To this final section, originally appearing as an article in *Casino and Sports*, can perhaps be traced the origins of Couponomy, of which science I am now regarded as the godfather. The term 'couponomy', meaning the extraction of wealth via coupon, was coined by the brothers Flowers, the foremost theorists and practitioners of the art.

SUPPLEMENT I

RULES AND CUSTOMS OF CASINO BLACKJACK

In the casino game of blackjack, the players do not compete among themselves, but rather, each gambles against the house, which is represented by a 'dealer' and a 'pit boss'. The dealer distributes cards to the players and himself and pays and collects all wagers. The pit boss supervises the game and is responsible for correcting any procedural errors.

The game is played with from one to eight ordinary 52 card decks. If more than two decks are used it is called 'multiple deck' blackjack and the cards are dealt from a dealing box (called a 'shoe') instead of being held in the dealer's hands.

In the play of the game, suits play no role. Aces may be valued as either one or eleven while jacks, queens, and kings count as ten points apiece in determining the players' or dealer's total. The other cards, ranked two through ten, count their face value for this purpose.

To begin the game, the player must make a wager, placing either currency or chips on the felt table in the specially marked circle corresponding to his seat. Then both the player and dealer are dealt two cards each. It is irrelevant whether the player's cards are exposed or not, but the dealer always has one card face up (called the 'up card') and one card face down underneath (called the 'down card'). It is part of the procedure and rules of the game that the down card not be visible to the player.

Blackjack

If the dealer's up card is a ten valued card, he will immediately look at his down card to determine if he has 'blackjack', which means an ace and any ten valued card. A blackjack (or 'natural' as it is also called) for the dealer cannot be beaten by the player, who can at best tie the dealer if he too has a blackjack, but otherwise loses. When the player has a blackjack and the dealer does not, the player not only wins, but is paid a bonus of 3 to 2 odds.

Insurance

When the dealer's up card is an ace he will also check underneath to see if he has a ten valued card for a blackjack, but only after asking the player if he wants 'insurance'. This insurance, best considered a side bet, is a wager offered to the player that the dealer does indeed have a ten valued card under the ace and hence a blackjack. It is paid at 2 to 1 odds and, since the player may only insure for up to half his bet, the result is that a winning insurance bet is paid by the player's own lost wager and the player appears to have gotten a tie. After any insurance bets have been decided, play continues.

The Settlement

When neither player nor dealer has blackjack, the player resolves his hand first and then the dealer his. The general principle for determining who wins is that

a) If the player 'breaks' (or 'busts', both of which mean to accumulate a total in excess of 21), then the dealer wins, whether the dealer breaks subsequently or not.

b) If the player doesn't bust, but the dealer does, then the player wins.

c) If the player and dealer have the same total, it is a tie (called a 'push') and no payoff is made.

d) Otherwise the hand with the higher total wins.

Hitting and Standing

The player achieves his final total by either 'standing', which means drawing no more cards, or 'hitting' (requesting another card from the dealer). The value of this drawn card is added to the player's current total and the decision as to whether to hit or stand is made again. If, in this fashion, the player's total exceeds 21 (a bust), then the player is obliged to turn in his cards right away, at which time his losing wager is collected. Otherwise,when the player desires no further cards, he will place his two original cards underneath his wager in the betting square, the cards being put face down by custom.

The dealer, however, has no choice in his hitting and standing activity and must proceed, when his turn comes after that of the players, by house rules which always require him to stand on a total of 'hard' 17 or more. A total is called hard if either it does not contain an ace or it exceeds 11 counting any aces which may be present as one; otherwise the total is called 'soft' and is determined by counting exactly one ace as eleven, any others as one. The house rules for the dealer's soft hitting and standing strategy will be either to 'always hit soft 17' or to 'stand on soft 17', and this will be posted on the table. Thus an ace and a three would be soft 14 and the dealer would draw again, but an ace and a seven would be soft 18 and the dealer would stand. Similarly the dealer would draw a card to a hard 16 which consisted of a ten, five, and an ace but stand with a ten, six, and an ace since this would be hard 17. When the dealer finishes he usually announces his total if he didn't bust and says 'over' or 'too many'* if he did break.

Pair Splitting

The player, but not the dealer, has the right to 'split' his original two cards if they are of the same denomination,

*It is Professor Griffin's contention that the dealer should say "Too much," since the reference is to the aggregate total rather than the discrete number of cards in the broken hand. His good natured and helpful inquiry, "Too many what?", always seems to elicit the same response from dealers: they never bust again.

such as a pair of eights. If he chooses to do this (he is not obliged to), he separates the two cards in front of him and puts up another, matching, wager. Then he proceeds to play two separate hands according to the previous prescription, even to the point of usually being able to split any subsequent pair achieved by receiving another card of the same denomination directly on one of the original paired cards. If this 'resplitting' occurs, he must again match his original wager. An exception occurs with split aces, to which can be drawn only one card each.

Doubling Down

If the player so chooses after observing his original two cards, he may double his bet and receive exactly one more card. By custom the player turns his original cards face up, puts out a matching bet, and receives another card face down, which explains the term 'doubling down.' When the player doubles down he forfeits the right to draw more than once.

SUPPLEMENT II

CARD COUNTING

Learning the basic strategy for blackjack is like learning to float in water; it enables you to survive. But if you want to *get somewhere* something additional is required. In this sense learning to count cards for playing blackjack is analogous to learning to swim.

A System

An extraordinarily simple and effective card counting system for blackjack was proposed years ago by Harvey Dubner. It is based on a categorization of the thirteen denominations into three separate groups:

> Low cards, [2,3,4,5,6], whose removal from the deck increases the basic strategist's expectation on subsequently dealt hands,
>
> High cards, [A,10,J,Q,K], whose removal from the deck decreases the basic strategist's expectation on subsequently dealt hands, and
>
> Middle cards, [7,8,9], whose removal is of little consequence to the basic strategy player.

The player *tracks*, or *counts*, the cards as they are removed from the deck by assigning the value +1 to each low card and −1 to each high card. The middle cards are treated as neutral in that they are assigned the value zero and ignored as they leave the deck.

After each shuffle the player begins with a mental count of zero. He increases his mental count by one every time he sees a low card removed and made unavailable for subsequent play. Similarly he decreases his count by one for each high card eliminated. As an example, suppose the first three hands dealt were as follows:

Player's Cards	Dealer's Cards	Count Before Hand	Count After Hand
5,7	6,4,J	0	$+1+0+1+1-1=+2$
K,8	10,A	+2	$-1+0-1-1=-1$
Q,3,9	2,7	−1	$-1+1+0+1+0=0$

Betting by the Count

If the pre-deal count is positive the basic strategy player may presume an advantage in a single deck game and should try to bet more money than usual. If the pre-deal count is negative he should presume a disadvantage and bet as little as possible.

Multiple deck games usually begin with about a half per cent disadvantage for the full pack, so it may be necessary to have a pre-deal count as high as +4 before the bet is raised. How high the count must be to justify an increased wager when playing against more than one deck will depend on many things, among them, how many cards remain unobserved.

A count functions as a sort of galvanometer. Positive deflections reflect an improvement in the player's prospects, negative ones suggest the situation is worse for the player than it would be with a full deck.

The basic strategy player's *change* in advantage (from whatever the full deck advantage or disadvantage is) can be estimated by multiplying the "running" (or current) count by 26.5% and then dividing by the number of unseen cards. Thus a count of +7 with 106 cards left provides evidence that the player's expectation has increased by $+7\,(26.5\%)/106 = 1.75\%$. If this were a four deck game with a full deck, first hand, disadvantage of .50%, then his instantaneous edge would be estimated as $1.75 - .50 = 1.25(\%)$ due to the +7 count with 106 cards remaining.

Varying Strategy by the Count

Although it's often complicated, and in many cases unproductive, the player can improve his playing of hands by occasionally changing the basic strategy in response to information provided by the count. Three frequently occurring and important variations in strategy of this nature are presented here as an illustration. Many others are possible.

1. The player should make an insurance bet *after* the first round of play if the average number of points per card left in the deck exceeds

.026 for single deck,
.046 for double deck, and
.055 for four decks.

The reason for this is that a high count suggests there may be enough extra tens left in the deck so that the 2 to 1 payoff for insurance will make it a profitable bet.

2. The player should stand on totals of 16 against the dealer's ten if the running count is zero or positive, regardless of the number of decks. A positive count, again, is indicative of more tens and fewer good drawing cards available. Thus the player busts his 16 more often than usual and may also not have helped himself if his resultant total is less than the dealer's increasingly probable total of 20.

3. The player should draw a card to 12 against a 4 if the running count is zero or negative. The suggestion here, with negative counts, is that there will be fewer tens to bust the player and more small cards to help the dealer make the hand with the 4 showing.

In all three examples it is assumed that the player has already included his own cards and the dealer's up card to adjust his running count before the decision is made.

Blackjack system books usually provide more extensive advice on how to vary bets and playing strategy as the count changes. The principles underlying their recommendations are, or should be, similar to those explained here.

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INDEX OF CHARTS AND TABLES

Peter Griffin teaches mathematics at an obscure college on the West Coast. A master of subtlety and disguise, he prefers his casino visits to be incognito. Scenes such as the below have struck terror into the hearts of Nevada casino bosses for the last ten years. Griffin's winnings are rumored to run into the hundreds. At the present time, Hollywood has no plans for a movie about his life.

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