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INTERMEDIATE POKER MATHEMATICS

MARK BOLLMAN

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Intermediate Poker Mathematics

Intermediate Poker Mathematics provides a fascinating collection of mathematical questions set in the diverse world of poker. While it is absolutely possible that a poker player will glean some insight that will improve their skill at the table, this book is not intended primarily as a players' strategy manual, but rather as a means of building up readers understanding of the mathematical concepts at play in the complex world of poker. Although the book is suitable for a general audience, it is formatted in the style of a textbook, with exercises included at the end of each chapter to help build understanding.

Features

- Written in an approachable style with minimal mathematical prerequisites beyond basic algebra and arithmetic
- Replete with engaging exercises and examples
- Wide-ranging exploration of multiple forms of poker beyond the more well-known varieties.

Mark Bollman is a Professor of Mathematics and chair of the Department of Mathematics & Computer Science at Albion College in Albion, Michigan, and has taught 120 different courses in his career. Among these courses is "Mathematics of the Gaming Industry," where mathematics majors carefully study the math behind games of chance and travel to Las Vegas, Nevada, in order to compare theory and practice. He has also taken those ideas into Albion's Honors Program in "Great Issues in Humanities: Perspectives on Gambling," which considers gambling from literary, philosophical, and historical points of view as well as mathematically. Mark has also authored *Mathematics of Keno and Lotteries, Mathematics of Casino Carnival Games, Mathematics of The Big Four Casino Table Games: Blackjack, Baccarat, Craps, & Roulette, and Basic Gambling Mathematics* by CRC Press.

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Intermediate Poker Mathematics

Mark Bollman

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For my Traditionally Unconventional friends in the Winter Camp Future Society.
1977–Doomsday

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Preface

When I read “Teaching a University Course on the Mathematics of Gambling”, by Stewart N. Ethier and Fred M. Hoppe, in the *UNLV Gaming Research & Review Journal*, I was struck by the accurate line

> Unfortunately, the mathematics of [Texas] hold’em is either trivial (e.g., the probabilities of making a hand at the turn or the river) or extremely complicated,

as when assessing the relative strengths of an AK hand vs. 88, which calls for consideration of thousands of possibilities [30].

Two thoughts came to mind:

1. **There’s more to poker than Texas hold’em.**

   As an illustration of the rich variety of poker games available in the 21st century, consider the Coach’s Game at Resorts World in Las Vegas. This game, run by Donald Shiflett Jr., is a mixed game offering the following 17 poker variants—none of them Texas hold’em, to say nothing of 5-card draw, 5-card stud, or other well-known poker games [68].

   - Archie
   - Badacey
   - Badeucey
   - Badugi
   - Badugi High-Low
   - Deuce to Seven Drawmaha
   - Deuce to Seven Triple Draw Lowball
   - Drawmaha 49
   - Drawmaha High
   - Drawmaha High-Dugi
   - Drawmaha Low-Dugi
   - Drawmaha Zero
   - Five Card Double Board Omaha 8/OB Ultimate (8/OB stands for “8 or Better”)
   - Limit Five-Card Omaha 8/OB
   - Razzdeucey (Three Cards Down)
   - Stud 8/OB (Three Cards Down)
   - Stud High-Low No Qualifier (Three Cards Down)

2. **There's probably a "middle band" of poker mathematics, not restricted to Texas hold'em, that merits exploration.**

   This would encompass poker mathematics questions beyond the simple—as illustrated by the opening chapters of the excellent book *Introduction to Probability with Texas Hold'em Examples* [130]—but short of the level of complexity represented, for example, in the equally fine book *The Mathematics of Poker* [21].

That middle band is what I've tried to capture here.

With that in mind, this book is perhaps best understood as a collection of interesting mathematical questions set in the diverse world of poker. While it is possible that a poker player may find some information here that improves his or her skill at the tables, this is not intended primarily as a players' strategy manual.

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I have been privileged for over a decade to collaborate with a fine roster of editors at Taylor & Francis, and this project has been no exception. Callum Fraser and Mansi Kabra have been an excellent team to work with in moving this book through the writing and production process.

I would like to thank the staff at the Center for Gaming Research at the University of Nevada, Las Vegas, where I have spent several short visits researching the history of poker that informs this book.

Some of the material in Section 5.6 is taken from my paper "When Does A Straight Beat A Full House?", presented at the 18th International Conference on Gambling and Risk-Taking in Las Vegas, Nevada, on 25 May 2023.

Chapter 1

Counting the Cards

A standard deck of playing cards consists of 52 cards, 13 cards in each of four suits. The 13 cards within each suit are denoted ace (A), 2 or deuce, 3, 4, 5, 6, 7, 8, 9, 10 (or T), jack (J), queen (Q), and king (K). Aces may, depending on the card game being played, be considered as either high or low. The suits are clubs (♣), diamonds (♢), hearts (♡), and spades (♠). Clubs and spades are black; hearts and diamonds are red.

Figure 1.1 shows a standard English deck. Other decks may change the design of certain cards, especially the face cards (jacks, queens, and kings) and the ace of spades.

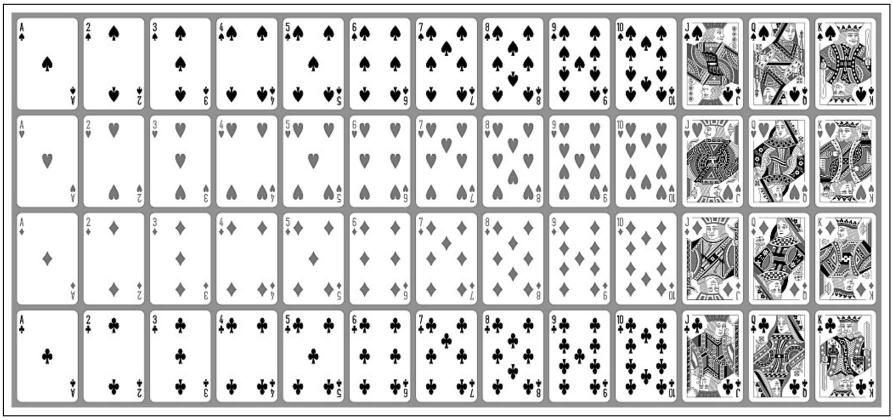

FIGURE 1.1: 52-Card English deck [41].

Some televised poker programs or online games color diamonds blue and clubs green for easy identification in a restricted space, but physical card decks with this coloring have not yet been successful. This may be because players fear that accidentally flashing a card during the deal would reveal too much information about its suit: a glimpse of the 9♢ from a standard deck reveals only that the card is a diamond or heart; if diamonds are blue, there is only one possibility for the suit if a flash of blue is seen.

Mike Caro, chief proponent since 1992 in the poker community for the four-color deck, has asserted that this argument against 4 colors is also a compelling argument in favor of having all 4 suits the same color [17].

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A standard deck may be enhanced by one or more *jokers*: extra cards not counted among the 52 and used as replacements for lost or damaged cards, or as wild cards (Section 1.6). Figure 1.2 shows a joker.

FIGURE 1.2: Sample joker from a standard card deck [111].

The design of jokers is not standardized as the shapes of the 4 suits are so different card manufacturers may have very different-looking jokers. The first joker appeared in America in the 1850s; the first card decks to include jokers were made by a Canadian company in 1887 [86]. Jokers are now a standard part of any deck of cards, though they are optional in gameplay.

1.1 Poker Hands

Whether played around dining room tables, in college dorm rooms, or at casino poker rooms like the one at the aptly-named Poker Palace in North Las Vegas, Nevada (Figure 1.3), the goal in many poker games is to make the best 5-card hand; however, the rules of the particular poker game define "best". The following 5-card hands, listed here from highest to lowest ranked, are commonly recognized.

1. The highest-ranked 5-card poker hand is a *royal flush*, which consists of the ace, king, queen, jack, and 10 of the same suit. If two players hold royal flushes in the same hand, they split the pot—suits have no standing in most poker games. Italian Poker (page 91) is an exception.

2. Royal flushes may also be thought of as the highest-ranking *straight flush*. A straight flush includes 5 cards of consecutive ranks in the same suit. Aces can count as high, which results in a royal flush, or low, as in $5432A\heartsuit$, the lowest-ranked straight flush. In a showdown between two straight flushes, the hand with the highest card wins. If two players have the same cards in different suits, they split the pot.

   The straight flush, or "sequence flush", to use its original name, is the newest widely recognized 5-card poker hand, arriving on the scene in the

FIGURE 1.3: The Poker Palace casino in North Las Vegas, Nevada [148].

late 19th century. The hand is mentioned in an 1875 book on poker [157], but an 1876 edition of *Hoyle's Book of Games*, the leading authority on rules of card games for decades, fails to mention either straight flushes or straights among its ranking of poker hands [42].

An 1876 poem, *The Game of Pokaire* by Bob Sea's Kenk, makes reference to the royal flush as a hand that beats 4 of a kind but does not mention other straight flushes [129].

3. *Four of a kind* is the third highest hand. These hands consist of all 4 cards of one rank together with an unmatched card.

   Prior to the recognition of the straight flush as a separate hand, a player holding 4 aces, or 4 kings and an ace, knew that his hand was unbeatable and could bet accordingly. This was thought by some gamblers and poker writers to be contrary to the spirit of gambling [87, 117]. Even if a player holds a royal flush, there is a very small chance, depending on the game, that one or more other players may also hold one, so the modern ranking of poker hands has no unbeatable hand.

4. A *full house* includes 3 cards of one rank and 2 cards of another rank.

   If two players hold full houses, the hand with the higher-ranked 3 of a kind wins the showdown, so 99922 beats $444AA$. The pair makes the hand a full house, of course, but otherwise it adds no value [87]. Since the pair is immaterial in the rankings, full houses are sometimes described briefly by the rank of the 3 of a kind, such as "aces full" for any hand of the form $AAAxx$.

In [118], Richard A. Proctor correctly notes that 3 aces and a pair of 3s beats 3 aces and a pair of 2s, but if both were to occur in the same hand of draw poker, there are larger questions to be answered than who holds the better cards.

5. A *flush* is a hand in which all 5 cards are of the same suit, but are not in sequence.

   When two or more players hold a flush, the hand with the highest card is the winner. An additional advantage of a 4-color deck is that flushes are easier to identify.

6. *Straights* have 5 cards in sequence, but not all in the same suit. A “Broadway straight” is the highest possible straight: $AKQJT$. As with straight flushes, an ace can be high or low, but straights are not allowed to go “around the corner”, counting the ace as both high and low, as in $KA234$. In some circles, hands like these were called “Hibernian straights”. An anonymous author writing under the name “Retired card sharp” in 1909 noted that “Round the Corner Straights are played in some of the Southern States”, where they rank as the lowest possible straight and are thus beaten by any conventional straight [120].

   As noted above under straight flushes, straights were not always accepted as a poker hand. An 1866 work by William B. Dick made mention of straights, or “rotations” but, while asserting that “straights are not considered in the game”, allowed that “they are played in some localities, and it should always be determined whether they are to be admitted at the commencement of the game” [27]. The book did, however, go on to support the recognition of the straight flush as the highest-ranking hand, citing the inadvisability of an unbeatable hand.

   In 1880, John Blackbridge agreed with the view that there should be no unbeatable poker hand and so favored straight flushes, but stopped somewhat short of fully embracing straights as a separate hand [8]. Blackbridge expressed hope that a Poker Congress might someday be convened to iron out these and other points of contention. By 1887, some authors were advocating that straights could only be played at poker with the unanimous consent of all players. That year, John W. Keller published *The Game of Draw Poker*, which addressed some of the confusion regarding straights by opining that straights “should always be played” [87]. Keller stated, arbitrarily yet correctly, that a straight should rank between a flush and 3 of a kind.

   While the Poker Congress envisioned by Blackbridge was never held, an effort was made in 1895 to lay out the “international code of laws” for poker and so resolve any local differences [147]. This set of rules included straights as a recognized hand and discouraged their omission as a hand type, and by the start of the 20th century, the straight and straight flush were acknowledged as canonical poker hands in most places. In 1945, Albert A. Ostrow's comprehensive guide to card games, *The Complete Card Player*, seemed to settle the matter while acknowledging the history of the straight, declaring "The straight is now officially recognized by poker players and must be considered as a legitimate combination" [112].

In a poker game played with 5-card hands dealt from a standard 52-card deck, any straight must contain either a 5 or a 10. This fact has some implications for stud poker games (Chapter 2) where each player can see some of their opponents' cards.

The straight, flush, full house, and straight flush (including royal flushes) are sometimes collectively called "complete" hands, since each card in the hand plays a part in its valuation. Prior to the correct mathematical ranking of straights among other poker hands, some authorities asserted that a straight should beat 3 of a kind simply because a straight is a complete hand while 3 of a kind isn't [155].

7. *Three-of-a-kind* hands are made up of 3 cards of one rank and 2 odd cards which match neither the 3 of a kind nor each other. In Texas hold'em (Chapter 4), 3 of a kind is usually called a *set*.

   We shall see that 3 of a kind loses to a straight. In 1881, a small book by Jack Abbott suggested that straights were by then "usually counted", but that players needed to agree at the outset of the game whether they beat 3 of a kind or merely beat 2 pairs [1]. Correct mathematical ranking of hands is not, of course, subject to a player vote.

8. *Two pairs* ranks next: a hand with two pairs of cards of different ranks and a fifth card not matching either pair.

   Two-pair hands are described by the rank of the higher pair, such as "9s up", for a pair of 9s and any lower pair. In a showdown between 2 two-pair hands, the highest pair wins. If both hands have the same higher pair, then the hand with the higher lower pair wins, for example, $KK88T$ beats $KK44Q$. Should the hands have the same pairs, the odd card determines the winner, so $7733A$ beats 77334.

9. A *pair* includes 2 cards of the same rank and 3 unmatched cards. Ties between pairs are broken by looking at the highest unpaired cards in each hand.

10. A *high card* hand consists of 5 unmatched cards, not in sequence, covering at least 2 suits. This is the lowest-ranked hand.

1.2 Elementary Probability Through Texas Hold'em

Consideration of questions that arise in the study of poker mathematics calls for familiarity with the language and methods of *probability*: the branch of mathematics that deals with uncertainty and randomness. Given an event A arising from a random experiment, we seek to assign a number $P(A)$, where $0 \leqslant P(A) \leqslant 1$, to the event as a measure of how likely it is to occur. $P(A)$ is called the *probability* of A. An impossible event is assigned probability 0, and an event that is certain to occur has probability 1. Values of $P(A)$ between 0 and 1 represent relative likelihood of A: an event with probability 0.3 is judged to be significantly less likely than an event with probability 0.8.

We will describe the simple laws of probability through a sequence of examples including several from one of the most popular forms of poker: *Texas hold'em*. In a hand of Texas hold'em:

- Each player is first dealt two face-down cards, or *hole* cards.
- Three cards, the *flop*, are then dealt face up to the table and are considered *community cards*, which may be used by each player to build a hand.
- Two additional community cards, the *turn* or *fourth street* and the *river* or *fifth street*, are dealt separately and face up.

At the end, each player makes his or her best 5-card hand from their 2 hole cards and the 5 community cards.

Definition 1.1. a. An *experiment* is a process that leads to some sort of outcome.

b. The set of all possible outcomes to an experiment is called the *sample space* and is denoted by $\mathbf{S}$.

c. An *event* A is any subset of $\mathbf{S}$.

For the probabilities that we will consider in this text, where the events and sample spaces are all finite, simple division suffices to perform the necessary calculations. We denote the number of elements in an event A by $\#(A)$. The formal definition of probability then involves the number of elements in A and the size of the *sample space*: the set of all possible outcomes to an experiment.

Definition 1.2. Let A be an event. The elements of A are *equally likely* if they all have the same chance of occurring.

For the first hole card in Texas hold'em, all 13 possible ranks are equally likely, since no cards have yet been dealt. Similarly, the card is equally likely to be any of the 4 suits.

Definition 1.3. Let A be an event in which every element is equally likely. The *probability* of A is defined as

$$P(A) = \frac{\#(A)}{\#(\mathbf{S})}.$$

It is important to specify that the elements comprising A be equally likely; if they are not, this simple formula is invalid.

Example 1.1. Some lottery players describe their probability of winning by saying "There are only two possible outcomes: either I win or I lose. That makes my chance of winning 50%". This reasoning, though it may be responsible for some of the success of state, provincial, and national lotteries, is incorrect since it assumes that winning and losing are equally likely. Tickets for lotteries such as Powerball have far more losing combinations than winning ones, making the probability of winning well below .5. ■

Addition Rules

Computing the probability of a more complicated event may be facilitated by breaking that event down into simpler events and carefully combining those probabilities.

Definition 1.4. Two events A and B are *mutually exclusive* if they cannot occur together; that is, if the occurrence of one implies that the probability of the other one occurring is 0.

Example 1.2. Considering the river card at hold'em, the two events

$$A = \{\text{The river is a heart}\}$$

and

$$B = \{\text{The river is a diamond}\}$$

are mutually exclusive, since each card has only 1 suit. The event

$$C = \{\text{The river is a red card}\}$$

is the event that either A or B occurs. ■

If two events are mutually exclusive, we can compute the probability that one of them occurs from their individual probabilities using a result called the *First Addition Rule.*

Theorem 1.1. ***(The First Addition Rule)*** *If A and B are mutually exclusive events, then*

$$P(A \text{ or } B) = P(A) + P(B).$$

Example 1.3. In Example 1.2, we have

$$P(A) = P(B) = \frac{1}{4}, \text{ and } P(C) = P(A) + P(B) = \frac{1}{2}.$$

■

If A and B are not mutually exclusive, a slightly more complicated formula can be used to calculate $P(A \text{ or } B)$.

Theorem 1.2. ***(The Second Addition Rule)*** *If A and B are any two events, then*

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

The additional term in the Second Addition Rule subtracts the probability that A and B occur together. If we apply the definition of $P(X)$ to each term, the third term counts the number of elements common to A and B, which have been counted twice: once as elements of A and once as elements of B. Subtracting $P(A \text{ and } B)$ ensures that each outcome is only counted once.

The First Addition Rule is a special case of the Second, for if A and B are mutually exclusive, then they cannot occur together, meaning that $P(A \text{ and } B) = 0$.

One obvious pair of mutually exclusive events is an event A and the contrary event that A does not occur.

Definition 1.5. Let A be an event. The event that A does *not* occur is called the *complement* of A and denoted A^C.

Example 1.4. If we draw one card from a standard deck and A is the event "An ace is drawn," the complement would be the event A^C: "The card drawn is not an ace". ■

We then have the following immediate corollary of the First Addition Rule called the *Complement Rule.*

Theorem 1.3. ***(The Complement Rule)*** *Let A be an event. Then*

$$P\left(A^C\right) = 1 - P(A).$$

The Complement Rule is sometimes useful in simplifying calculations, where it can reduce a great number of computations to one.

Example 1.5. Let A be the event that the river is a 2. The probability of A^C, that the river is not a 2, can be computed by calculating the 12 individual probabilities that the card is a 3, 4, 5, ..., Q, K, A, and adding them, or by reasoning that

$$P\left(A^C\right) = 1 - P(A),$$

which is easier to compute. The probability that the river is a 2 can be computed by counting the number of 2s in sight and doing some simple arithmetic; calculating the 12 complementary probabilities requires accounting for all of the other ranks. ■

Independent Events and Conditional Probability

Definition 1.6. Two events A and B are *independent* if the occurrence of one has no effect on the occurrence of the other one.

By specifying that A and B are independent, we are asserting that the probability of one occurring is not affected by whether or not the other one has occurred.

Example 1.6. We draw 1 card from a standard deck. Let A be the event "The card is a spade" and B be the event "The card is a 4". We have

$$P(A) = \frac{13}{52} = \frac{1}{4}.$$

If we are told that event B has occurred, then

$$P(A) = \frac{1}{4},$$

since there are four 4s in a deck, and only 1 is the 4♠. The fact that B has occurred has no effect on $P(A)$.

If we turn this around and are told that the event A has occurred, the probability that the card is an 4 is

$$P(B) = \frac{1}{13},$$

since of the 13 spades, there is only one 4♠.

If we consider a draw from the full 52-card deck with no knowledge about event A, we find that

$$P(B) = \frac{4}{52} = \frac{1}{13},$$

the same value obtained above. Taken together, these 2 calculations comprise sufficient evidence that A and B are independent events. ■

Two events that are mutually exclusive (page 7) are explicitly *not* independent, since the occurrence of one eliminates the chance of the other occurring. Moreover, two events that are independent, as in Example 1.6, cannot be mutually exclusive.

During the progress of a hand of any form of poker, the cards that may be dealt depend on the cards that have already been dealt, and so successive cards are not independent.

Example 1.7. If the flop in Texas hold'em consists of the $A\diamondsuit$, $Q\diamondsuit$, and $2\diamondsuit$, the probability of another diamond on the turn decreases, since the composition of the deck has changed. Conversely, if the flop shows $J\heartsuit$ $9\heartsuit$ J♣, the probability of a diamond on the turn increases, since the number of cards of other suits has gone down. ■

However, it is a fundamental principle of gambling mathematics that *successive trials of random experiments are independent.* Where poker is concerned, this means that a player's successive hands, with the deck shuffled between hands, are independent. The cards that a player holds in one hand have no relation to the cards dealt on the previous hand, nor do they have any influence on the hands which follow. Poker players may have winning or losing streaks, but provided that the cards are thoroughly shuffled, these streaks are determined by chance and player skill, not to any connection among successive hands. If the cards are not thoroughly shuffled, or if a cheating dealer in a home game carefully collects the cards and stacks the deck in an order that advantages either himself or a collaborating fellow player, then the hands dealt in the next round may be similar to the hands in the round just completed. Carefully shuffling the cards and cutting the deck before each deal will help to ensure independence among hands.

In video poker, where virtual cards are shuffled and dealt by a computer, successive hands are truly independent. Though a player may have occasional winning or losing streaks, there is no connection among successive hands.

If A and B are independent events, it is a simple matter to compute the probability that they occur together using a theorem called the *Multiplication Rule.*

Theorem 1.4. ***(Multiplication Rule)*** *If A and B are independent events, then*

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

Informally, the Multiplication Rule states that we can find the probability that two successive independent events both occur by multiplying the probability of the first by the probability of the second. The rule can be extended to any finite number of independent events: the probability of a sequence of n independent events is simply the product of the n probabilities of the individual events.

Example 1.8. In the video poker game *Five Deck Poker* (page 301), each card in a 5-card hand is dealt from a separate 52-card deck. Since each deck is handled individually, the probability of getting an ace on any one card is not affected by whether or not any aces have been dealt previously. Successive cards are, therefore, independent. The probability of drawing an ace from a fresh deck is $\frac{4}{52}$, so the probability of a 5-ace hand is then

$$\left(\frac{4}{52}\right)^5 = \frac{1}{371,293}.$$

■

Example 1.9. Suppose that you hold 4♢ 5♢ 6♢ 7♢ J♠ A♡ after the flop and the turn, and call all bets in the hope of filling in your hand to a straight, flush, or straight flush on the river. Fifteen of the 46 unseen cards will complete your hand: the 3♢ and 8♢ that give you a straight flush, the 6 other 3s and 8s that result in a straight, and the 7 other diamonds that complete a flush, so the probability of success is $\frac{15}{46}$. In the language of hold'em, this hand is said to have 15 *outs.* Outs are considered in more detail beginning on page 116.

If we look at this situation recurring over 2 successive hands, the hands are independent, and so the probability of completing the hand twice is

$$\left(\frac{15}{46}\right)^2 = \frac{225}{2209} \approx .1063.$$

■

If the events A and B are not independent, we will need to generalize Theorem 1.4 to handle the new situation. This generalization requires the idea of *conditional probability.* We begin with some examples.

Example 1.10. If we draw one card from a standard deck, the probability that it is a queen is $\frac{4}{52} = \frac{1}{13}$. If, however, we are told that the card is a face card, the probability that it's a queen is $\frac{4}{12} = \frac{1}{3}$—that is, additional information has changed the probability of our event by allowing us to restrict the sample space. If we denote the events "The card is a queen" by Q and "The card is a face card" by F, this last result is written $P(Q \mid F) = \frac{1}{3}$ and read as "the (conditional) probability of Q given F is $\frac{1}{3}$". ■

The fundamental idea here is that more information can change probabilities. If we know that the event A has occurred and we're interested in the event B, we are now looking not for $P(B)$, but $P(B \text{ and } A)$, because only the part of B that overlaps with A is now possible. With that in mind, we have the following formula for conditional probability:

Definition 1.7. The *conditional probability* of B given A is

$$P(B \mid A) = \frac{P(B \textit{ and } A)}{P(A)}.$$

This formula divides the probability that both events occur by the probability of the event that we know has already occurred. Note that if A and B are independent, we immediately have $P(B \mid A) = P(B)$, since then $P(B \text{ and } A) = P(A) \cdot P(B)$. This is one case where more information—in this case, the knowledge that A has occurred—does not change the probability of B occurring.

Example 1.11. In Example 1.10, we have

$$P(Q \mid F) = \frac{P(Q \textit{ and } F)}{P(F)}.$$

$P(Q \textit{ and } F) = \frac{4}{52}$, and $P(F) = \frac{12}{52}$. Dividing gives

$$P(Q \mid F) = \frac{4/52}{12/52} = \frac{1}{3},$$

as we calculated earlier. ■

Example 1.12. Suppose that the dealer inadequately protects the bottom card of the deck, and a quick flash reveals that it's the $A♠$—which is easy to identify because many card manufacturers use an ornate design for their $A♠$ with an enlarged spade (Figure 1.4). This card is effectively out of play, and

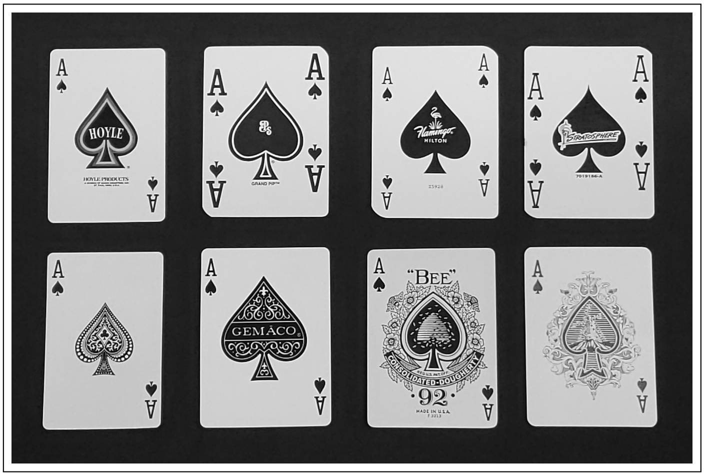

FIGURE 1.4: A variety of $A♠$s, with ornate symbols indicating the manufacturer or casino.

so any calculation involving aces or suits should take that into account.

- The probability that your first hole card is an ace has dropped from $\frac{4}{52} = \frac{1}{13}$ to $\frac{3}{51} = \frac{1}{17}$.
- The probability that your first hole card is a heart has risen, from $\frac{1}{4}$ to $\frac{13}{51}$.

■

As with the addition rules, we can state a second, more general, version of the Multiplication Rule that applies to any two events—independent or not—and reduces to the first rule when the events are independent. This more general rule simply incorporates the conditional probability of B given A, since we are looking for the probability that both occur.

Theorem 1.5. ***(General Multiplication Rule)*** *For any two events A and B, we have*

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A).$$

We see that independent events make calculations easier. This being the case, we shall at times assume that certain events are independent for ease of computation, provided that that assumption doesn't introduce an unacceptable level of inaccuracy to the final result—see Example 1.41 for an illustration.

Example 1.13. In a hand of Texas hold'em, suppose that your hole cards are $K\spadesuit$ $T\spadesuit$ and that the flop contains the 7♠ and 5♠, giving you a flush draw. The probability that you miss the flush on the turn but complete it on the river can be computed using the General Multiplication Rule with

A = Fail to complete the flush on the turn.
B = Complete the flush on the river.

We then have $P(A) = \dfrac{38}{47}$, since we want the turn to be one of the 38 non-spades remaining in the deck, and then $P(B \mid A) = \dfrac{9}{46}$: the chance of drawing one of the 9 remaining spades from the 46 cards that remain after the turn. The General Multiplication Rule gives

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A) = \frac{38}{47} \cdot \frac{9}{46} \approx .1582,$$

close to 1 chance in 6. ■

1.3 Combinatorics

The probability formulas described above are collected in Appendix A, on page 333. In these basic formulas, we are usually not as interested in listing every element of A or of $\mathbf{S}$ as we are in knowing how many elements the sets contain. *Combinatorics* is the branch of mathematics that studies techniques for counting sets—of numbers, playing cards, or any other items of interest. When considering mathematical questions that arise in poker games, we often find ourselves considering the number of ways in which several events can happen in sequence, such as the draw of several cards to a hand. If the events of interest are independent and we know the number of ways that each individual event can happen, elementary combinatorics tells us that simple multiplication can be used to find the answer. This is expressed in the *Fundamental Counting Principle.*

Fundamental Counting Principle

Theorem 1.6. ***(Fundamental Counting Principle)*** *If there are n independent tasks to be performed, such that task T_1 can be performed in m_1 ways,* *task* T_2 *can be performed in* m_2 *ways, and so on, then the number of ways in which all* n *tasks can be performed successively is*

$$N = m_1 \cdot m_2 \cdot \ldots \cdot m_n.$$

That the Fundamental Counting Principle (FCP) is a reasonable result can be easily seen by testing out some examples with small numbers and listing all possibilities—for example, when rolling two six-sided dice (abbreviated 2d6), one red and one blue, each die is independent of the other and can land in any of six ways. By the Fundamental Counting Principle, there are $6 \cdot 6 = 36$ ways for the two dice to fall, and this may be confirmed by writing out all of the possible outcomes. It may be useful to think of the dice as being different colors, so that ⚂ ⚃ is a different roll from ⚃ ⚂, even though the numbers showing are the same. Table 1.1 shows all of the possible sums.

TABLE 1.1: Sample space of 36 equally-likely outcomes when rolling 2d6.

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
⚁	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
⚂	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
⚃	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
⚄	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
⚅	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

A special case of the Fundamental Counting Principle arises when we consider the number of ways N to arrange a set of n elements, with no repetition allowed, in different orders. The first element may be chosen in n ways, the second in $n - 1$, and so on, down to the last item, which may be chosen in only 1 way. The total number of orders for a set of n elements is thus $N = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. This number is given a special name, n *factorial*.

Definition 1.8. If n is a natural number, the *factorial* of n, denoted $n!$, is the product of all of the positive integers up to and including n:

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n.$$

$0! = 1$, by definition.

It is an immediate consequence of the definition that $n! = n \cdot (n-1)!$. Factorials get very big very fast. $4! = 24$, but then $5! = 120$ and $6! = 720$. $10!$ is greater than 3 million.

Example 1.14. A standard deck of cards may be rearranged into 52! different orders. There are 52 choices for the first card, then 51 choices for the second card, and so on until we have only 2 choices for card #51 and one choice for the 52nd and final card.

52! is approximately 8.0658×10^{67}. This number is so large that if you thoroughly shuffle a deck of cards, it is highly likely that you are arranging them in an order that has never occurred before in the history of the 52-card deck. ■

Permutations

In Example 1.14, the order in which the cards appear matters. This is called a *permutation* of a set: in this case a deck of cards.

Definition 1.9. A *permutation* of r items from a set of n items is a selection of r items chosen so that the order matters.

For example, ABC is a different choice of three alphabet letters than CBA. It should be noted that "order" may appear in several forms. One way to determine whether or not order matters in making a selection is to ask if different elements of the selection are being treated differently once they are chosen.

When considering permutations, we are usually interested in how many permutations of a given set exist, not a list of the many ways that even a small set may be rearranged. The number of permutations of size r drawn from a given set of size n, which is denoted ${}_nP_r$, can be easily calculated using the following theorem.

Theorem 1.7. *The number of permutations of r items chosen from a set of n items is*

$${}_nP_r = \frac{n!}{(n-r)!}.$$

Proof. There are n ways to select the first item. Once an item is chosen, it cannot be chosen again, so the second item may be chosen in $n-1$ ways. There are then $n-2$ items remaining for the third choice, and so on until there are $n-r+1$ numbers remaining from which to choose the rth and final term. By the Fundamental Counting Principle, we have

$${}_nP_r = n \cdot (n-1) \cdot \ldots \cdot (n-r+1).$$

Multiplying the right-hand expression by $1 = \frac{(n-r)!}{(n-r)!}$ gives

$$\begin{aligned} {}_nP_r &= n \cdot (n-1) \cdot \ldots \cdot (n-r+1) \cdot \frac{(n-r)!}{(n-r)!} \\ &= \frac{n \cdot \ldots \cdot (n-r+1) \cdot (n-r) \cdot \ldots \cdot 3 \cdot 2 \cdot 1}{(n-r)!} \\ &= \frac{n!}{(n-r)!}, \end{aligned}$$

as desired. □

We see that, in Example 1.14, we calculated the number ${}_{52}P_{52}$.

Poker games which involve permutations are scarce: if you're dealt 4 aces and a king, it's bad form to complain that the king was dealt to you fourth and broke up the block of consecutive aces. It's a high-ranking hand regardless of the order in which the cards arrived. Badugi (page 188), a 4-card poker game where the goal is to get a low hand with all 4 suits present, is one game where order matters in building a hand.

Example 1.15. Another poker game where the order of the cards matters is a game variation that offers a bonus if a video poker player receives a *sequential* royal flush—the ace, king, queen, jack, and 10 of the same suit in either ascending ($TJQKA$) or descending ($AKQJT$) order. In a 1998 promotion at Baldini's Casino in Sparks, Nevada, the promotion was offered to anyone cashing their paycheck at the casino, who received a free spin on a special video poker machine. This machine offered a million-dollar payout to any player dealt a royal flush in one of these ways [102].

Receiving a dealt royal flush is hard enough ($p = 4/2,598,960$; see page 19); to get the cards in order is considerably more difficult. The number of possible arrangements of 5 cards out of 52, where the order matters, is

$${}_{52}P_5 = 311,875,200.$$

Of these hundreds of millions of arrangements, there are 2 orders and 4 suits that lead to royal flushes, so the FCP tells us that there are only 8 sequential royals. The probability of a dealt sequential royal flush, therefore, is

$$\frac{8}{311,875,200} = \frac{1}{38,984,400}.$$

The last denominator is close to the 2023 population of Canada, making the probability of winning the million dollars about the same as randomly choosing a resident of Canada and getting the mayor of Windsor, Ontario.

This was a free promotion, so there was no risk to players, but there was also very little risk to the casino offering this game. There were other prizes offered; the most commonly won prize was a free beer [102]. ■

Distinguishable Permutations

A special case of the permutation formula arises when we consider the number of ways to arrange a collection which contains some repeated items. For example, in rearranging the letters in the word MASSACHUSETTS, permutations that differ only by scrambling the 4 copies of the letter S are indistinguishable. If we are only interested in permutations that are visually different, we need to divide out the number of ways to arrange each set of repeated letters—here, the S's, A's, and T's.

In general, the number of *distinguishable permutations* of a collection A containing n_1 interchangeable items of type 1, n_2 of type 2, and so on up to n_k items of type k, where $N = n_1 + n_2 + \cdots + n_k$, is

$$\frac{N!}{n_1! \cdot n_2! \cdots n_k!}.$$

There are then

$$\frac{13!}{4! \cdot 2! \cdot 2! \cdot (1!)^5} = 64,864,800$$

distinguishable permutations of the word MASSACHUSETTS, $\frac{1}{96}$ of the total number of 13! ways to rearrange 13 different alphabet letters.

Some card games use nonstandard decks where cards are repeated. Pinochle uses a 48-card deck with 2 copies of each card from 9s through aces in all 4 suits. Pokara (page 268), a poker-like game offered at the Gardens Casino in Hawaiian Gardens, California, uses a deck consisting of multiple copies of the tens through aces in a standard deck. Counting hands dealt from these decks may call for the formula above.

The 48 cards in a pinochle deck, where each of 24 cards is represented twice, can be shuffled into

$$\frac{48!}{(2!)^{24}} \approx 7.3993 \times 10^{53}$$

distinguishable orders. While this is far fewer arrangements than the 52! that are possible with a standard 52-card deck, there are still enough possibilities that a thorough shuffle of a pinochle deck will almost certainly leave the cards in an order that has never been seen before.

Combinations

Most of the time when gambling, we are not so concerned about the order of events, as when a hand of cards is dealt. A 5-card draw poker player may arrange the 5 cards in some particular order as he or she may prefer, perhaps sorted by ranks or by suits, but changing the order of the cards does not change the rank of the hand. For counting these arrangements, we are interested in *combinations.*

Definition 1.10. A *combination* of r items from a set of n items is a subset of r items chosen without regard to order. The number of such combinations is denoted $\binom{n}{r}$, which is read as "n choose r". An alternate notation for this number is ${}_nC_r$.

Here, ABC and CBA are identical combinations, as they are subsets of the alphabet consisting of the same three letters. The different order is not a concern here. If the elements of a selected subset are receiving the same treatment once selected, then the selection is a combination.

Theorem 1.8. *The number of combinations of r items chosen from a set of n items, where $n \geqslant r$, is*

$$\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}.$$

If $n < r$, we define $\binom{n}{r}$ to be 0.

Proof. We begin with the formula for the number of permutations:

$${}_nP_r = \frac{n!}{(n-r)!}.$$

Since we are looking for combinations, two permutations that select the same elements and differ only in their order are identical to us. Any combination of r elements from a set of n can be rearranged into ${}_rP_r = r!$ different orders that are interchangeable, by the Fundamental Counting Principle. Dividing out these duplicate permutations of the same combination gives

$$\binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)! \cdot r!},$$

as desired. □

Example 1.16. Since the order in which the cards are dealt does not matter, combinations are used to count poker hands. The number of 5-card poker hands that can be dealt from a standard 52-card deck is

$$\binom{52}{5} = \frac{52!}{47! \cdot 5!} = 2,598,960.$$

■

Example 1.17. Contract bridge deals 13 cards to each of 4 players, using the full deck. The number of possible 13-card bridge hands is then

$$\binom{52}{13} = \frac{52!}{39! \cdot 13!} = 635,013,559,600.$$

■

Example 1.18. *Euchre* is a card game, popular in the American Midwest, that deals 5-card hands from a 24-card deck consisting of the 9s through aces in all 4 suits. The number of euchre hands is

$$\binom{24}{5} = \frac{24!}{19! \cdot 5!} = 42,504.$$

■

Example 1.19. $\binom{14}{4} = 1001$, so close to 1000 that the National Basketball Association uses this set of combinations as the foundation of its draft lottery. The teams that miss the playoffs in a given season are entered into a lottery to determine the order in which teams draft new players coming into the league. Each team is assigned a certain number of combinations based on their regular-season records, with lower-performing teams assigned more combinations and thus a higher chance of winning the first pick. The three teams with the poorest records each receive 140 combinations, and so have a 14% chance of winning the first pick. The lottery team with the best record is assigned 5 combinations, giving a 0.5% chance of the top pick. Fourteen numbered ping-pong balls are thoroughly mixed and four are drawn. The combination 11-12-13-14 corresponds to no team; if the draw produces 11-12-13-14, the balls are returned to the mixer and a new combination is drawn.

Once the first pick is assigned, the draw is repeated for the next 3 positions, with the provision that no team may win more than one pick in the top 4 unless previous trades entitle it to another team's pick. Lottery picks 5–14 are then assigned to the remaining teams in the reverse order of their regular-season records. ■

1.4 Counting 5-Card Hands

The formula for combinations, together with the Fundamental Counting Principle, allows us to count the number of poker hands of the different ranks described at the start of this chapter. In the analysis that follows, we will occasionally impose an artificial order on the cards in a hand in order to organize our thinking. This does not contradict the fact that the order in which the cards are dealt to a poker hand does not matter.

Royal flushes are easy to count: there is 1 in each suit, for a total of 4, since there is no choice of the ranks. To compute the probability of a royal flush, we divide 4 by the number of possible 5-card hands, yielding

$$P(\text{Royal flush}) = \frac{4}{2,598,960} = \frac{1}{649,740}.$$

For **straight flushes**, we focus on the lowest-ranked card in the flush. Any card from ace through 9 can be lowest; if the lowest card is a 10, the straight flush is a royal flush, and we have already counted those. Once this card is identified, the other 4 are completely determined. Multiplying by 4 to account for all suits gives $4 \cdot 9 = 36$ possible straight flushes.

There is a total of 40 royal flushes and straight flushes, a number which we will subtract from future calculations when counting flushes and straights since these 40 hands will not be counted as ordinary flushes or straights.

There are 13 ranks to choose from for **four-of-a-kind** hands. Once the rank is chosen, there is only 1 way to choose the cards. We then have 48 choices for the odd card, making $13 \cdot 48 = 624$ ways to draw four of a kind.

For **full houses**, we count the three of a kind and the pair separately, and then use the Fundamental Counting Principle to multiply the results. There are 13 choices for the rank of the triple, and the cards may then be chosen in $\binom{4}{3} = 4$ ways. Another way to think of this is that there are 4 ways to choose the card of the selected rank that is *not* in the hand. Moving to the pair, there are 12 ways to pick the rank, since we cannot duplicate the rank of the triple, and then $\binom{4}{2} = 6$ ways to choose the cards. Combining everything gives $(13 \cdot 4) \cdot (12 \cdot 6) = 3744$ full houses.

It should be noted here that the factors of 13 and 12 are more explicitly described as $\binom{13}{1}$ and $\binom{12}{1}$; in each case, we are making 1 choice from a set. As a general principle, we have

$$\binom{n}{1} = \frac{n!}{(n-1)! \cdot 1!} = n$$

for all natural numbers n.

Moving on to flushes and straights: These are two hands whose relative ranks are often confused. On the face of it, there doesn't seem to be much difference between "5 cards of one suit, not in sequence" and "5 cards in sequence, not all of the same suit". Careful counting will confirm the hand order listed in Section 1.1.

For **flushes**, the count is

$$4 \cdot \binom{13}{5} - 40 = 5108,$$

where we subtract the sum of royal flushes and straight flushes.

The number of **straights** is computed as we did when counting straight flushes: by starting with the lowest card, which can be any ace through 10. Once the lowest card is selected, there are 4 ways to pick each remaining card going up in sequence. As with flushes, we subtract 40 to remove royal and straight flushes, giving

$$40 \cdot 4^4 - 40 = 10,200,$$

so there are nearly twice as many straights as flushes.

In counting **three-of-a-kind** hands, it is necessary to exclude the 4368 four of a kinds and full houses from the count. Rather than subtract that number at the end of the calculation, we shall build that restriction into our analysis. Since the hand contains three cards of one rank, it cannot also be a flush or straight. We go card by card: There are 13 ranks to choose from for the triple and then $\binom{4}{3} = 4$ ways to select the cards. The remaining two cards must be of two different ranks and must not be the fourth card of the triple. We can choose the ranks in $\binom{12}{2} = 66$ ways, and there are then 4 choices for each card. The total number of three of a kinds is then

$$13 \cdot 4 \cdot 66 \cdot 4^2 = 54,912.$$

As with three of a kind, hands scoring as **two pairs** must also be carefully counted to exclude full houses and four of a kinds. They number

$$\binom{13}{2} \cdot \binom{4}{2}^2 \cdot 11 \cdot 4 = 123,552,$$

where the first two factors select the two different ranks and two cards from each rank. Once this choice has been made, there are 11 ranks for the fifth card and 4 cards to choose within that rank.

Hands containing only **one pair** number

$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1,098,240.$$

The meaning of the factors can be derived by analogy with the calculations for three of a kind and two pairs. We note that this number is divisible by 13; the number of pairs of a given rank is just $\frac{1}{13}$ of 1,098,240: which is 84,480.

Example 1.20. In many video poker games, and some carnival games where payoffs for pairs differ based on the rank of the paired cards, this number can be very useful. For a video poker game where the lowest paying hand is a pair of jacks, there are $4 \cdot 84,480 = 337,920$ one-pair hands that pay off. ■

Finally, **high card** hands can be counted most easily by subtracting the total of all hands listed above from 2,598,960. This gives 1,302,540, so the probability of a hand that does not contain at least a pair is

$$\frac{1,302,540}{2,598,960} = \frac{1277}{2548} \approx .5012,$$

just over 50%.

Alternately, we may compute this number directly by considering rank and suit choices in a high card hand. There are $\binom{13}{5} - 10$ ways to choose the ranks of the 5 cards, where we subtract 10 to remove the straights from $A2345$ through $TJQKA$. The suits of 5 cards that do not form a flush can be picked using the Fundamental Counting Principle in $4^5 - 4$ ways; subtracting 4 removes the flushes. The total number of high card hands is found by multiplication:

$$\left[\binom{13}{5} - 10\right] \cdot \left(4^5 - 4\right) = 1277 \cdot 1020 = 1,302,540$$

again.

Table 1.2 collects the frequencies of the 2,598,960 possible 5-card poker hands that we have derived here.

TABLE 1.2: Frequencies of 5-card poker hands.

Hand	**Count**
Royal flush	4
Straight flush	36
Four of a kind	624
Full house	3744
Flush	5108
Straight	10,200
Three of a kind	54,912
Two pairs	123,552
Pair	1,098,240
High card	1,302,540

Certain facts about the distribution of 5-card poker hands are evident from this table:

- The median 5-card hand is a high-card hand—more than 50% of all hands are lower than a pair.
- The 90th percentile of all hands is hand #259,896 from the top. This is on the high end of the one-pair hands.
- The 99th percentile is hand #25,990—somewhere in the block of 3 of a kinds. Less than 1% of all 5-card hands are a straight or higher.

Master Formulas

We can combine some of these counting techniques into a single master formula for counting hands in a 52-card deck [22]. The number of ways to draw exactly h sets of i cards of the same rank and j sets of k cards of the same rank among a hand of m cards, where $hi + jk \leqslant m$, is

$$\binom{13}{h}\binom{4}{i}^h \cdot \binom{13-h}{j}\binom{4}{k}^j \cdot \binom{13-h-j}{m-hi-jk}\binom{4}{1}^{m-hi-jk}.$$

In this formula:

- The factor $\binom{13}{h}\binom{4}{i}^h$ counts the number of ways to choose h ranks from 13 and then to choose i cards from the 4 cards in each suit. The exponent h counts the number of sets of i cards.
- Once this choice has been made, there are $\binom{13-h}{j}\binom{4}{k}^j$ ways to choose j ranks from the $13 - h$ ranks that remain, and then k cards of each chosen rank.
- At this point, there are $13 - h - j$ ranks left untouched, and we want to choose $m - hi - jk$ of them to fill out the hand with singleton cards drawn from the 4 suits.

For example, when counting full houses in a 5-card hand, we have

$$h = 1,\ i = 3,\ j = 1,\ k = 2,\ \text{and } m = 5.$$

This formula encompasses the calculations for four-of-a-kind hands, full houses, three of a kinds, two-pair hands, and pairs.

Counting straights and flushes, including straight flushes, involves the following additional formulas:

$$\text{Straight flushes:} \quad 4(15 - m)$$

This number includes royal flushes.

$$\text{Flushes:} \quad 4\binom{13}{m} - 4(15 - m)$$
$$\text{Straights:} \quad (15 - m)\binom{4}{1}^m - 4(15 - m).$$

The number 15 arises in these formulas because there are 13 ranks in a deck and the ace can rank both high or low in a straight or straight flush. The number of possible lowest cards in an n-card straight is then $13 - (m - 2) = 15 - m$, since the top $m - 2$ ranks, other than the ace, are not low enough to fit $m - 1$ cards in sequence above them. If $m = 5$, then $15 - m = 10$, which indicates that any card from the ace through 10 can be the lowest card in a straight or straight flush.

1.5 Lowball Hands

In some versions of poker, the winner is the player with the lowest hand, not the highest; these games are collectively called *lowball* games. Some lowball games disregard straights and flushes in the race to the bottom while others rank these hands in their natural order and thus render them undesirable to gamblers.

Other games split the pot between the players with the lowest and highest hands. In such a game with more than 5 cards dealt per hand, players may choose different cards from their hands to make their best high and low hands. A player who has both the highest and lowest hands at the showdown is said to *scoop* the pot.

Example 1.21. In a 7-card game, a player holding

$$A\heartsuit\ A\diamondsuit\ Q\heartsuit\ 7\spadesuit\ 6\heartsuit\ 4\heartsuit\ 2\heartsuit$$

could form a heart flush as high hand and $7642A$ as a 7-high low hand. (Aces frequently count only as low in lowball poker.) This choice would stand a very good chance at winning both high and low, scooping the pot. ■

A number of lowball games have gained favor in casinos and card rooms.

- In *Ace to 5* or *California* lowball, straights and flushes are not counted as high hands, and aces always rank low. The lowest possible hand is $5432A$, which is called a *wheel* or *bicycle*. Ace to 5, a draw game, is often played with the 53-card deck including a wild joker that can be used as any card the player doesn't already hold but needs to make their lowest possible hand.
- *Deuce to 7* lowball is another draw game, where straights and flushes count as high hands. Furthermore, aces always rank high, so the lowest possible hand—from which the game takes its name—is 75432 of mixed suits. In some places, this game is called *Kansas City Lowball*.
- *Razz* (Section 2.3) is 7-card stud played for low. Razz usually uses the hand and card ranking rules in Ace to 5.

The rules of lowball games vary somewhat; some hybrid games in which straights and flushes are disregarded also require that aces rank high, so the lowest hand would be 65432.

Unless the game includes a small number of players, lowball games are very difficult to win if your best low 5-card hand contains even a pair, since it is likely that an opponent holds a simple high-card hand. In a small game, say one with 3 or 4 players and no draw where a pair might well qualify as low hand, players would do well to establish at the start whether or not a pair of aces is lower than a pair of 2s.

If straights and flushes are not counted as high hands, there are 1,317,888 hands which do not contain a pair, three of a kind, or four of a kind and thus count as high-card hands for the purposes of lowball poker. To count the number $F(x)$ of high-card hands with high card x, where aces are low and the jack, queen, and king have ranks 11, 12, and 13 respectively, we have the following formula:

$$F(x) = 4 \cdot \binom{x-1}{4} \cdot 4^4,$$

for $5 \leqslant x \leqslant 13$. Here, the first 4 counts the number of choices for the high card and $\binom{x-1}{4}$ counts the number of ways to pick the ranks of the 4 lower cards. Each rank may then be chosen in 4 ways (suits), leading to the factor of 4^4.

$F(x)$ is tabulated in Table 1.3.

TABLE 1.3: Lowball poker: Number of high-card hands with highest card specified. Straights and flushes count as high-card hands, and aces rank low.

High card	**Hands**
King	506,880
Queen	337,920
Jack	215,040
10	129,024
9	71,680
8	35,840
7	15,360
6	5120
5	1024

If straights and flushes count as high hands and aces are allowed to rank low, then the lowest possible hand of the 1,302,540 high card hands counted in Section 1.3 is $6432A$ of at least 2 suits. For each high card, there are $4^5 = 1024$ straights, including straight flushes, which must be subtracted from $F(x)$ if straights and flushes are counted as high hands. Additionally, there are $4 \cdot \binom{x-1}{4} - 4$ flushes with high card x, where the -4 at the end removes straight flushes. Collecting everything gives

$$\sum_{x=5}^{13} \left[F(x) - 1024 - \left(4 \cdot \binom{x-1}{4} - 4 \right) \right] = 1,303,560.$$

There are 1020 extra high-card hands collected here; this is because we have included the 1020 ace-high straights (not including the 4 royal flushes, which are king-high flushes and thus high hands) that are counted as king-high non-straights in lowball.

Some games require that the winning low hand be 8-high or lower. Table 1.3 shows that there are only 57,344 qualifying low hands if a joker is not used, so the probability that a 5-card hand qualifies as low is approximately .0221, just over 2%.

1.6 Wild Cards

In the opinion of the author, any wild game should be barred from the United States by an act of Congress, if necessary, and only permitted in some foreign countries, including parts of Arkansas.

—General Service Company, [55].

The 1930 opinion of the General Service Company notwithstanding, some poker games use one or more *wild cards*: cards that may be assigned any value the player wishes. An advantage to the use of wild cards is that players tend to hold more highly-ranked hands, and thus may be more likely to play longer and bet more money, which increases the size of the pot. This is somewhat flawed player logic, though, since if everyone has access to wild cards, everyone's hand can be higher, which means that playing and betting strategies must change to take this into account. However, in a game of stud poker where many player cards are dealt face up and there's no opportunity to exchange cards, an exposed (or *open*) wild card may depress the action as players can plainly see that they've been beaten.

Wild cards may be one or more jokers added to the deck, or may be cards within the 52-card deck—for example, all 4 deuces—that are designated as wild. The number of wild cards can vary greatly with time and place.

- 1 wild card is typically a single joker, creating a 53-card deck. In 19th-century France, this lone wild card was called the *mistigris*.
- Some games using 2 wild cards simply add 2 jokers and play with 54 cards. Other games, especially informal home games, may designate the one-eyed jacks ($J\spadesuit$ and $J\heartsuit$) as wild and play with a 52-card deck. Mathematically, these are very different decks.
- 3 wild cards, the $A\diamondsuit$, $J\clubsuit$, and $9\diamondsuit$, were found in the game of *brag*, an ancestor of poker described on page 164.
- 4 wild cards can be achieved by designating all deuces as wild, as is done in some video poker games.
- 5 jokers were added to the deck in a game variation described in 1916 [38].

- A game called *Paresis Poker* used 6 wild cards, two jokers and the four deuces. Gamblers were warned against this variant in 1910, when it was cited as an unnecessary addition to a game that had recently been perfected by the acceptance of straights, jack pots (page 48), and some common-sense rules about betting [7].
- *Woolworth Draw*, named for the American 5- and 10-cent store, designates all 5s and 10s as wild, resulting in 8 wild cards.
- *Dr Pepper* (page 106) calls all 10s, 2s, and 4s wild, so there are 12 wild cards.

When a joker is used, the game rules may restrict its use to completing a straight or a flush, and count it as an ace otherwise. In this capacity, the joker is sometimes called a *bug*.

Example 1.22. With this restriction on the joker, a hand such as Joker $T\heartsuit$ $T\diamondsuit$ 9♠ 9♣ cannot be raised to a full house, since the bug cannot be used as a 10 or 9 here. The joker is interpreted as an ace. ■

Consider a 53-card deck including one fully wild joker.

- The highest-ranked poker hand is now 5 of a kind, which can be made in 13 ways. Any 5 of a kind constitutes an unbeatable hand, since there is only 1 joker. This might be undesirable. Historically, there was resistance in some quarters to five-of-a-kind hands. Additionally, since the joker has no suit, five of a kind necessarily requires duplicating a card in the player's hand.

  Sandy Griswold, sports editor of the *Omaha World-Herald*, held in his weekly column for many years that "The joker does not strengthen four of any denomination" [61, 62]. Griswold used the example that 4 aces plus the joker should not beat a royal flush, missing the fact that if one player holds all 4 aces and the lone joker, no one else at the table can hold a royal flush.
- Royal flushes occur in two types: 4 natural royal flushes without the joker, and 20 royal flushes containing the joker, for a total of 24.
- Counting straight flushes requires a bit more care. Let W denote a wild joker, and consider the straight flush $\heartsuit 34567$. Replacing any of the 4 highest cards with a joker generates a jokered version of the same hand, but in the hand $W\heartsuit 4567$, the joker will be read as the $8\heartsuit$ and the hand played as the higher straight flush $\heartsuit 45678$.

There are 36 natural straight flushes. Following the example above, we see that in each, 4 of the 5 cards can be replaced with the joker to generate a jokered version of the same straight flush, and multiplying shows that there are $36 \cdot 4 = 144$ new straight flushes involving the joker. Adding back the 36 natural straight flushes gives a total of 180.

If the joker can only be used as a bug, the number of straight flushes, flushes, and straights do not change from the counts using a fully wild joker, but other hands change in frequency. Five of a kind can only be 5 aces, and full houses are less common, as noted in Example 1.22 where many 2-pair hands with the bug in addition cannot rise to a full house.

Example 1.23. Consider 3-of-a-kind hands. When holding a natural 3 of a kind plus the bug, the 5th card cannot be an ace which would make the hand a full house. Nor can the 3 of a kind in-hand be 3 aces. These hands number

$$12 \cdot \binom{4}{3} \cdot 44 = 2112.$$

There are

$$\binom{4}{2} \cdot \binom{12}{2} \cdot 4^2 = 6336$$

ways to hold 3 of a kind consisting of 2 aces, the bug, and 2 nonmatching cards.

Adding these numbers to the 54,912 three-of-a-kind hands without a wild card gives a total of 63,360 three of a kinds in a 53-card deck with a bug. ■

Table 1.4 shows the hand frequencies for 5-card hands dealt from a 53-card deck including a wild joker and from a deck where the joker is only used as a bug. These figures include the 2,598,960 hands without a joker or bug.

TABLE 1.4: Poker hand frequencies: 53-card deck with 1 wild joker and with a joker used as a bug [13].

	Frequency	
Hand	**With joker**	**With bug**
Five of a kind	13	1
Royal flush	24	24
Straight flush	180	180
Four of a kind	3120	828
Full house	6552	4368
Flush	7804	7804
Straight	20,532	20,532
Three of a kind	137,280	63,360
Two pairs	123,552	138,600
One pair	1,268,088	1,215,024
High card	1,302,540	1,418,964

Note that the median hand has risen from a high-card hand without the bug to a low 1-pair hand when the bug is included.

Rule Changes

If poker is played with a 53-card deck including a single fully wild joker, Table 1.4 shows that three of a kind becomes more common than two pairs. If a player is dealt, say, W 9♠ 9♢ 7♢ 6♣, the joker could be regarded as a third 9 or as a second 7, and counting the hand as three 9s is better than calling it two pairs because of how hands are ranked. In general, there are no two-pair hands that include a wild card, keeping the number of such hands at the 123,552 computed on page 21. In a 53-card deck with a fully wild joker, there are

$$\binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{2} \cdot 4^2 = 82,368$$

hands of the form $Wxxyz$ that could be valued at either 3 of a kind or 2 pairs.

Proper adherence to the frequencies of the various hands would then require that the ranking of hands in 53-card poker should be rearranged to place two pairs above three of a kind. In practice, of course, this would simply result in players opting to define a hand including a pair and a joker as two pairs rather than three of a kind, and so negate the mathematical correctness of the new hand ordering.

Several attempts have been made to change the rules of poker with a wild card that respect the different order of hand frequencies. Gambling expert John Scarne suggested the following two possible changes to the rules of poker, for players interested in following correct mathematics [127]:

1. In a hand consisting of a pair plus a joker, the joker cannot be called wild and must be valued as a third unmatched card. This rule change means that no two-pair hands or three of a kinds contain a wild card.

   The rule retains the order of the two hands while allowing certain no-pair hands such as W 2♡ 7♡ 9♡ Q♡ to rise to higher hands with the joker.

2. Alternately, in a hand with pair of 8s through aces plus a joker, the joker may be used to raise the hand to three of a kind. A pair of 2s through 7s plus a joker must be called two pairs.

   Applying this rule means that

   $$7 \cdot \binom{4}{2} \cdot \binom{12}{2} \cdot 4^2 = 44,352$$

   hands containing a joker and a pair advance to three of a kind, and

   $$6 \cdot \binom{4}{2} \cdot \binom{12}{2} \cdot 4^2 = 38,016$$

   hands must be called two pairs. Adding these to the number of natural hands of those ranks gives 99,264 three of a kinds and 161,568 two-pair hands. The order of the two hands is preserved.

Under either one of these alternate hand-ranking schemes, three of a kind is less common than two pairs. In suggesting these rule changes, Scarne was well aware that his hybrid schemes might be mathematically correct, but would be very unlikely to catch on for casual play.

Example 1.24. It is shown in Table 1.4 that 3 of a kind outranks 2 pairs when the joker is restricted to use as a bug. There are 63,360 three-of-a-kind hands in a 53-card deck with a bug, and 138,600 two-pair hands in that deck. ■

A later examination of the mathematics by John Emert and Dale Umbach, considering one, two, or four wild cards, showed that if wild cards are allowed in poker, it is not possible to rank the hands so that more valuable hands occur less frequently without restrictions on how wild cards may be valued [29]. No matter how the various hands are ranked, there will always be a way to value hands containing wild cards so that a more common hand is ranked higher than one that is less common.

Example 1.25. As an example of the challenges that arise when playing poker with wild cards, we will show that in a 54-card deck with 2 wild jokers, there is an equal number of full houses and four-of-a-kind hands.

Natural hands of both types were counted beginning on page 20: there are 624 four-of-a-kind hands and 3744 full houses without jokers.

To form a four-of-a-kind hand with 1 joker, the hand must be $Wxxxy$, where x and y are different card ranks—which could also be called a full house, but isn't because four of a kind beats a full house with a standard deck. There are 2 choices for the joker, 13 for the rank of the triple, $\binom{4}{3} = 4$ ways to pick the cards of the triple, and 48 ways to choose the odd card of rank y. Multiplying gives 4992 one-joker four-of-a-kinds.

A full house with a single joker looks like $Wxxyy$, and the joker is used as the higher rank of x and y. These hands number $2 \cdot \binom{13}{2} \cdot \binom{4}{2}^2 = 5616$.

Hands with 2 jokers become four-of-a-kind hands if the hand is $WWxxy$, with $x \neq y$. There is only 1 way to select the jokers, and then $13 \cdot \binom{4}{2} \cdot 48 = 3744$ ways to fill out the rest of the hand.

There are no full houses with 2 jokers, since any such hand would contain a natural pair of xs and would thus be played as four of a kind. These hands look like $WWxxy$, and there are $13 \cdot \binom{4}{2} \cdot 48 = 3744$ of them.

Adding everything up gives 9360 of each type of hand. ■

The implications of this result for gameplay are left to individual poker players. It is important for all players to agree on how jokers may be used when playing poker with wild cards.

The Wild Card Rule

Further analysis by Kristen Lampe proposed the *Wild Card Rule* as a way to preserve hand rankings while incorporating one or more jokers into the game [89].

> **Wild Card Rule**: A poker hand containing one or more wild cards is assigned its *second highest* possible rank, using the standard poker hand rankings as an ordering.

It is vital that players using the Wild Card Rule agree to do so before playing poker with a wild card.

Example 1.26. The following are illustrations of how the Wild Card Rule works in practice.

1. In a 53-card deck with 1 wild joker, a hand only counts as 3 of a kind under the Wild Card Rule if it contains no wild cards. Three of a kind is a possible hand valuation with a joker only if the hand also contains a natural pair. The hand W 8♢ 8♠ T♡ 6♡ can be valued as 3 of a kind, 2 pairs, or 1 pair. The Wild Card Rule would call this 2 pairs. This is how any hand of the form $Wxxyz$ will play.

   A hand of the form $Wxxxy$ can be scored as either 3 of a kind, a full house, or 4 of a kind. Since the full house ranks second, the hand is called a full house.

2. W K♣ K♠ T♣ T♡ can be read as a full house or 2 pairs. Two pairs is the correct value according to the Wild Card Rule.

3. Any hand of the form $Wxyzw$, with no flush or straight possibilities, cannot be promoted to one pair and must be valued as a high-card hand.

4. Consider the hand W W K♣ J♣ J♠, with 2 wild cards. This hand can be 4 of a kind, a full house, 3 of a kind, 2 pairs, or 1 pair. The second-highest option is a full house: kings over jacks.

5. There are no 5 of a kind hands using the Wild Card Rule. Since 5 of a kind ranks highest, no hand containing a wild card can have 5 of a kind as its second-highest valuation. A desirable corollary to the Wild Card Rule is that there is no unbeatable hand, even with a wild card in play.

■

Example 1.27. If a game uses 2 or more wild cards, the hand W W J♢ 6♢ 3♣ can be counted as 3 of a kind, 2 pairs, 1 pair, or high card. The Wild Card Rule directs that this hand play as 2 pairs.

If the 3♣ is replaced by the 3♢, the new hand can also be counted as a flush, and so is elevated to 3 of a kind. ■

One unexpected challenge to the Wild Card Rule arises with a hand like $W\ K\diamondsuit\ 9\diamondsuit\ 5\diamondsuit\ 3\diamondsuit$. The possible assignments of W give a flush, a pair, or a high-card hand. The Wild Card Rule ranks this hand as a pair of kings. If the game is draw poker with a single joker, a player might be tempted to discard the joker in the hope of improving the hand to a natural flush [89].

By discarding the wild card and drawing, the following hands, shown with their probabilities, are possible:

$$P(\text{Flush}) = \frac{9}{48} = \frac{3}{16}.$$
$$P(\text{Pair}) = \frac{12}{48} = \frac{1}{4}.$$
$$P(\text{High card}) = \frac{27}{48} = \frac{9}{16}.$$

The probability of improving this pair of kings by discarding the joker is only $\frac{3}{16}$. There is a $\frac{1}{16}$ chance that the hand remains a pair of kings, and the probability that the hand declines in value is $\frac{3}{4}$.

A better strategy might be to discard 3 cards, keeping the king and joker, or just to hold the joker and draw 4 new cards.

One-Eyed Jacks

Another option for introducing wild cards into poker is to designate cards within the standard deck as wild, such as one-eyed jacks: the $J\spadesuit$ and $J\heartsuit$, which traditionally appear in profile on the cards. See Figure 1.5.

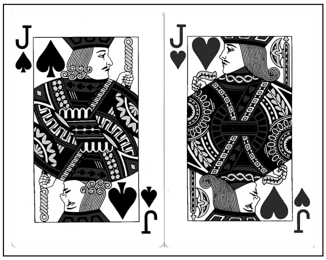

FIGURE 1.5: One-eyed jacks. [34].

Designating these two cards as wild changes the probability calculations, since counting most hands requires accounting for the wild cards and for the diminished number of natural jacks.

Suppose that a 5-card poker hand contains j wild jacks and k non-wild jacks. The number of such hands is

$$N(j,k) = \binom{2}{j} \cdot \binom{2}{k} \cdot \binom{48}{5-j-k}.$$

Table 1.5 shows the values of $N(j,k)$ for $0 \leqslant j,k \leqslant 2$.

TABLE 1.5: Number $N(j,k)$ of 5-card hands with j wild jacks and k non-wild jacks.

j \ k	**0**	**1**	**2**
0	1,712,304	389,160	17,296
1	389,160	69,184	2256
2	17,296	2256	48

Consider the case where $j = 1$ and $k = 1$; a hand holding 1 wild jack and 1 non-wild jack. There are

$$2 \cdot 2 \cdot \binom{48}{3} = 69,184$$

such hands. These take the form $WJxyz$, where none of x, y, or z is a jack. There will be 2 ways to choose the wild jack and 2 ways to choose the non-wild jack.

We shall assume here that the Wild Card Rule is *not* in use. While the value of the hand depends on x, y, and z, certain hands are impossible: 5 of a kind, full house, 2 pairs, and high card. Regarding full houses, a hand holding 3 of a kind, non-jacks ($WJxxx$), will promote to 4 of a kind, and a hand with a pair in addition to the jack ($WJxxy$) will play as 3 of a kind.

- A royal flush must be in diamonds or clubs, which is determined by the non-wild jack. The other 3 cards must be chosen from the 10, queen, king, and ace of that suit; there are $\binom{4}{3} = 4$ ways to choose them. There are 2 ways to pick the suit of the jack and 2 ways to pick the wild card, which gives 16 royal flushes.

- Straight flushes must also be in diamonds or clubs. There are 3 possibilities: $789TJ, 89TJQ$, and $9TJQK$. The number of straight flushes starting with a given rank is the same as the number of royal flushes: 16. Accordingly, there are $3 \cdot 16 = 48$ straight flushes.

- To draw 4 of a kind requires a hand of the form $WJxxx$: 3 of a kind plus the 2 jacks. Counting gives

$$2 \cdot 2 \cdot 12 \cdot \binom{4}{3} = 192$$

four-of-a-kind hands.

- There are
$$2 \cdot 2 \cdot \binom{12}{3} - 64 = 816$$
flushes, which must again be $\diamondsuit$s or $\clubsuit$s. The 64 straight and royal flushes are removed at the end.

- Straights allow the sequence $TJQKA$ along with the 3 possibilities under straight flushes. For a straight whose lowest natural card is 7–9, there are
$$2 \cdot 2 \cdot \binom{3}{2} \cdot \left(4^3 - 1\right) = 756$$
possibilities; multiplying by 3 to account for the choices for the low card gives 2268. The factor $4^3 - 1$ counts the suits, removing the possibility that the 3 cards are of the same suit, which would create a straight or royal flush. A hand such as $W89TJ$ will count the wild jack as a queen and play as 8-low rather than 7-low. Straights starting with a 10 allow for the wild jack to serve as any card in the sequence other than the jack, so there are
$$2 \cdot 2 \cdot \binom{4}{3} \cdot \left(4^3 - 1\right) = 1008$$
$TJQKA$ straights. Adding up gives 3276 straights.

- Three of a kinds must look like $WJxxy$, with the wild card used as a third x. These number
$$2 \cdot 2 \cdot 12 \cdot \binom{4}{2} \cdot 44 = 12,672.$$

- Hands with one pair are $WJxyz$ with x, y, and z all of different ranks and not forming a straight with the W and J. Their suits must not all be the same as the suit of the J. Counting gives
$$2 \cdot 2 \cdot \binom{12}{3} \left(4^3 - 1\right) - 3212 = 52,228$$
pairs. In this calculation, the factor 4^3-1 counts the suits while removing all flushes, and subtracting 3212 at the end eliminates the straights that are not flushes.

Example 1.28. A hand holding both wild jacks will be 3 of a kind or higher, and cannot be a full house unless the Wild Card Rule is used. A full house with a pair of wild jacks requires another pair in hand, and so will be played as 4 of a kind. The number of 4 of a kinds of the form $WWxxy$ is
$$12 \cdot \binom{4}{2} \cdot 44 = 3168.$$

If, in addition, the hand holds 1 of the 2 other jacks, taking the form $WWJxy$, the hand is 4 of a kind only if $x = y$, since it cannot be 4 jacks. There are

$$2 \cdot 12 \cdot \binom{4}{2} = 144$$

such hands. ■

The Ultimate In Wild Card Poker

> *[I]n a game with enough wild cards five aces could be good only for a tie.*
>
> —Hugh Zachary, [159].

At the extreme end of the wild-card spectrum, the smallest deck for which 5 of a kind is the most common hand, with probability .3266, is a 125-card deck containing 73 wild jokers. If, instead, 110 jokers are added to create a 162-card deck, the probability of 5 of a kind is

$$\frac{\sum_{k=1}^{4}\left[\binom{110}{k} \cdot 13 \cdot \binom{4}{5-k}\right] + \binom{110}{5}}{\binom{162}{5}} \approx .5033,$$

making this deck the smallest one for which the probability of 5 of a kind exceeds 50%. The probability of a high-card hand drops to .0015 [32].

In this massive deck, the least probable hand is two pairs. As we saw with a single wild card, if the standard 5-card hand rankings remain in force, there are no two-pair hands that use any of the wild cards. It follows that the number of two-pair hands remains 123,552 and their probability is

$$\frac{123,552}{873,642,672} \approx 1.414 \times 10^{-4},$$

approximately one hand in 7071.

1.7 Random Variables

Many questions of probability in poker are simply answered by computing $P(A)$ for a single identified event A. Others may have cause to consider multiple competing events with their associated probabilities. Questions of this latter type most commonly arise when we consider carnival games based on poker (Chapter 7), some of which include optional side bets whose payoffs are based on the strength of the player's cards. Discussion of this type of question is facilitated by introducing the concept of a *random variable*.

Definition 1.11. A *random variable* X is a quantity whose value is determined by chance.

Random variables often arise as outcomes of an experiment. Frequently, though not always, a random variable is a numeric quantity.

Example 1.29. a. Draw a single card from a standard deck and record its suit. This is a case where the values of X are non-numeric: $X = \clubsuit, \diamondsuit, \heartsuit$, or $\spadesuit$.

b. Deal a 5-card poker hand and let X be the number of aces it contains. X is an integer in the set $\{0, 1, 2, 3, 4\}$.

c. In 1000 completed video poker hands, let X count the number of hands of a pair of jacks or better, which is frequently the lowest winning video poker hand. Here, X can theoretically be any integer value from 0–1000.

■

Each possible value a of a random variable X has a probability $P(X = a)$ associated with it. If we collect these values across all values of a, we have a complete description of the likelihood of X attaining any of its possible values. This is called a *probability distribution function* or *PDF.*

Example 1.30. In Example 1.29a, the PDF is $P(X = a) = \frac{1}{4}$ for any value of the suit variable a. A PDF like this, where each possible value of the random variable is equally likely, is called a *uniform* PDF. ■

Example 1.31. In Example 1.29b, the number of aces in a 5-card poker hand is given by the following PDF:

$$P(X = a) = \frac{\binom{4}{a} \cdot \binom{48}{5-a}}{\binom{52}{5}},$$

where $0 \leqslant a \leqslant 4$.

In this formula, $\binom{4}{a}$ counts the number of ways to choose a aces from a deck containing 4 aces, $\binom{48}{5-a}$ counts the number of ways to choose the remaining $5 - a$ non-aces from the 48 non-aces in the deck, and $\binom{52}{5}$ is the number of possible 5-card hands. This function is tabulated in Table 1.6.

■

TABLE 1.6: Probability of being dealt a aces in 5 cards.

a	$P(X = a)$
0	.6588
1	.2995
2	.0399
3	.0017
4	1.847×10^{-5}

Binomial Distribution

Example 1.29c describes a random variable X counting the number of 5-card video poker hands of a pair of jacks or better. Most forms of video poker involve draw poker, which allows a gambler to discard some initial cards and draw replacements. The probability that a single hand reaches jacks or better depends on player choice in selecting discards. There is a best possible strategy for discarding and drawing, which is determined in part by the game's pay table. One common set of game rules, combined with the best strategy for those rules, gives a probability $p \approx .4484$ of a winning hand [133]. What is the probability that a player wins r times in a sequence of 1000 hands, which represents the number of hands that the best video poker players can play per hour?

Solving this problem is facilitated by introducing the concept of a *binomial experiment*.

Definition 1.12. A *binomial* experiment has the following four characteristics:

1. The experiment consists of a fixed number of successive identical trials, denoted by n.

2. The trials are independent.

3. Each trial has exactly two outcomes, denoted *success* and *failure*.

   In practice, it is often possible to amalgamate multiple outcomes into a single category to get down to two. For example, in the video poker example above, we can collect all of the possible losing hands into a single outcome—if you lose your bet, it really doesn't matter much what your hand was.

4. The probabilities of success and failure are constant from trial to trial. We denote the probability of success by p and the probability of failure by q, where $q = 1 - p$.

   The Complement Rule is easily applied here, since we have defined the problem so that there are only 2 outcomes.

Definition 1.13. A random variable X that counts the number of successes of a binomial experiment is called a *binomial* random variable. The values n and p are called the *parameters* of X. We may indicate the parameters of a binomial random variable by saying that X is $B(n, p)$.

The experiment described at the beginning of this section meets the four listed criteria and is therefore a binomial experiment. If we let X denote the number of winning hands in 1000 hands, then X is $B(1000, .4484)$.

If we change the experiment to "Start playing video poker, and let the random variable X be the number of hands required to win exactly 20 times," then the new experiment is not binomial. Since the number of trials is no longer fixed at the outset, criterion 1 is no longer true.

If X is a binomial random variable with parameters n and p, the formula for $P(X = r)$ can be derived through the following three-step process:

1. Select which r of the n trials are to be successes. This can be done in $\binom{n}{r}$ ways, as the order in which we select the successes does not matter.

   If we think of the trials as a row of n boxes, each to be designated "success" or "failure," what we're doing here is determining which r of the n boxes are successes.

2. Compute the probability of these r trials resulting in successes. Since the trials are independent, this probability is p^r.

3. We must now ensure that there are *only* p successes. This is done by assigning the outcome "failure" to the remaining $n - r$ trials. The probability of this many failures is $(1 - p)^{n-r} = q^{n-r}$.

Multiplying these three factors together gives the following result, called the *binomial formula*:

Theorem 1.9. *If X is a binomial random variable with parameters n and p, then*

$$P(X = r) = \binom{n}{r} \cdot p^r \cdot q^{n-r} = \binom{n}{r} \cdot p^r \cdot (1 - p)^{n-r}.$$

Example 1.32. Using the probability $p \approx .4484$ described above for the chance of a winning video poker hand, we can construct the PDF for X, where $0 \leqslant r \leqslant 1000$.

$$P(X = r) = \binom{1000}{r} \cdot p^r \cdot (1 - p)^{1000-r}.$$

The probability of winning exactly half of these hands is

$$P(X = 500) \approx 1.1933 \times 10^{-4},$$

approximately 1 chance in 8380. ■

Expected Value

The notion of *expected value* is fundamental to any discussion of random variables and is especially important when those random variables arise from a game of chance. The expected value of a random variable X is, in some sense, an average value, or what we might expect in the long run if we were to sample many values of X.

The common notion of "average" corresponds to what statisticians call the *mean* of a set of numbers: add up all of the numbers and divide the sum by how many numbers there are. For a random variable X, this approach requires some fine-tuning, as there is no guarantee that a small sample of values of X will be representative of the range of possible values. Our interpretation of average will incorporate each possible value of X together with its probability.

Definition 1.14. Let X be a random variable with a given probability distribution function $P(X = x)$. The *expected value* or *expectation* $E(X)$ of X is computed by multiplying each possible value for X by its corresponding probability and then adding the resulting products:

$$E(X) = \sum_x x \cdot P(X = x).$$

The notation used here does not indicate the limits of the indexing variable x, as is customary with sums; this is because those values may not be a simple list running from 0 or 1 to some n. When written this way, we should take the sum over all possible values of the random variable X. This expression may be interpreted as a standard arithmetic mean drawn from an infinitely large random sample. If we were to draw such a sample, we would expect that the *proportion* of sample elements with the value x would be $P(X = x)$; adding up over all values of x gives this formula for $E(X)$.

We may abbreviate $E(X)$ as E when the random variable is clearly understood.

Example 1.33. Example 1.31 gives the PDF $P(X = a)$ for the number of aces in a 5-card poker hand. An "average" hand contains

$$E(X) = \sum_{a=0}^{4} a \cdot P(X = a) = \frac{5}{13}$$

aces. ■

Often, a random variable X is used to describe the outcome of a round or hand of a casino game. The expected value of X is then taken as a "typical" outcome of one round.

Example 1.34. Suppose you wager \$1 on the number 0 at a European roulette table. This bet pays off at 35–1, and has probability $\dfrac{1}{37}$ of winning.

If we define the outcome of this bet by X, then X can take on the two values 35 and –1. The expected value of X is

$$E(X) = (35) \cdot \frac{1}{37} + (-1) \cdot \frac{36}{37} = -\$\frac{1}{37} \approx -\$0.0270.$$

In the long run, you can expect to lose 2.7¢ every time you make this bet. While you will never lose exactly 2.7¢ on a single spin, it is highly likely that your net loss after many spins will be approximately 2.7¢ times the number of spins. ■

If a random variable is binomial, computing its expected value is simple.

Theorem 1.10. *If X is a binomial random variable with parameters n and p, then $E(X) = np$.*

Another way to say this is that the average number of successes is the number of trials multiplied by the probability of success on a single trial.

Example 1.35. In Example 1.32, the number of winning hands in 1000 hands of video poker was shown to be a binomial random variable with parameters $n = 1000$ and $p \approx .4484$. The expected number of winning hands in a run of 1000 hands is then

$$E(X) = np \approx 448.4.$$

■

Definition 1.15. A game or wager is *fair* if its expected value is 0.

Example 1.36. Suppose that you toss a coin with a friend, with the loser paying \$1 to the winner. Your probability of winning is ½, hence the expected value of this game is

$$E = (1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0,$$

and the game is fair. ■

If a game is fair, neither side has an advantage, and in the long run, each player should expect to break even.

Casinos don't offer very many fair games. One exception is the Double Up bet on some video poker machines, which is described on page 328.

House Advantage

A concept related to expected value is the *house advantage (HA)* associated with a game of chance. The HA provides a measure of how much of the money wagered by a gambler or gamblers will, in the long run, be kept by the game operator.

Definition 1.16. The *house advantage* of a game with a wager of N and payoffs given by the random variable X is $-\frac{E(X)}{N}$.

If the expectation is negative, as it is in virtually every casino game, the HA will be positive. The house advantage of a game is frequently expressed as a percentage of the original wager. In Example 1.34, where the expected value of a $1 bet is –$.0270, the HA is 2.70%.

House advantages in casino games range from nearly 0% in blackjack played with certain rules and perfect player strategy to over 20% in some versions of keno.

Example 1.37. *Players Choice* was a proposed casino table game that combined 5-card poker with elements of a lottery. Six 5-card poker hands were dealt face down. Players made a wager and could then choose one of the hands. If that hand turned out to be the highest-ranked hand when the cards were turned over, the bet paid off at 4–1. A player could choose to make multiple wagers and select multiple hands.

Assuming no tied hands, the probability of selecting the best hand is 1 in 6, so the expected value of a $1 bet is

$$E = (4) \cdot \frac{1}{6} + (-1) \cdot \frac{5}{6} = -\$\frac{1}{6} \approx -\$.1667.$$

The house advantage of Players Choice was then 16.67%. Since this was too high for the game to be viable on its own, bonuses were paid for certain high hands [50]. Bonuses started at 1–1 for 2 pairs and ran up to 500–1 for a royal flush.

A look at Table 1.2 on page 22 reveals that the probability of a dealt hand qualifying for a bonus was only 7.63%. ■

State, provincial, and national lotteries, whose mission is in part to raise funds for various good works including public education, typically have a house edge that runs from 40–50%. From 1982–1985, the Michigan State Lottery offered a daily poker-based game called simply "Card Game". A $1 Card Game ticket bore three randomly-selected playing cards, and a nightly computerized drawing from a separate complete deck produced 2 community cards which were added to every gambler's ticket.

On August 28, 1982, the first drawing produced the $J♣$ and 8♠ as community cards, but the game had a rocky first year. In an early version of the Y2K bug, Card Game's computers were unable to change the dates on the tickets from 1982 to 1983 when the year changed. The game was suspended in early January 1983 and did not return until June [80, 85].

Card Game payoffs were awarded, using Table 1.7, for any completed 5-card hand of 2 pairs or better. While many winning possibilities were immediately ruled out once the ticket was printed, any Card Game ticket could win one of the lowest 2 prizes if the community cards came in right. The use of 2 decks raised the number of possible 5-card tickets from 2,598,960 to

$$\binom{52}{3} \cdot \binom{52}{2} = 29,304,600.$$

TABLE 1.7: Michigan Lottery's Card Game pay table.

Poker hand	**Payoff**
Royal flush	\$50,000
5 of a kind	\$5000
Straight flush	\$4000
4 of a kind	\$50
Full house	\$20
Flush	\$15
Straight	\$10
3 of a kind	\$3
2 pairs	Free ticket

Of these, 25,989,600 contain 5 different cards; this number is the number of 5-card hands drawn from a single deck multiplied by $\binom{5}{3} = 10$, the number of ways to choose 3 of them to comprise the cards on the printed ticket.

Due to the 2-deck configuration, the hand types in Table 1.7 are not mutually exclusive, though players were paid only for the highest combination on their tickets. It would be possible for a ticket to hold a 2-pair flush: for example, with $K\diamondsuit$ $Q\diamondsuit$ $8\diamondsuit$ on the printed ticket and community cards of $8\diamondsuit$ and $Q\diamondsuit$. There are

$$4 \cdot \binom{13}{3} \cdot \binom{3}{2} = 846$$

ways to draw a 2-pair flush.

In addition to introducing 5 of a kind as a possible winning hand, using a separate deck for the final 2 cards changed the probabilities of winning hands. To win the top prize with a royal flush, 2 events must happen:

1. The ticket has to contain 3 cards to a royal flush. This has probability

$$\frac{4 \cdot \binom{5}{3}}{\binom{52}{3}} = \frac{2}{1105}.$$

2. The drawing then needs to deliver the 2 remaining cards of the royal. The probability of this event is

$$\frac{1}{\binom{52}{2}} = \frac{1}{1326}.$$

Since the two draws are independent, we can compute the probability of a royal flush by multiplying these two probabilities, which gives

$$P(\text{Royal flush}) = \frac{1}{732,615},$$

approximately $\frac{8}{9}$ of the probability of drawing a royal flush from a full single deck.

The Card Game pay table suggests that a royal flush is considerably rarer than 5 of a kind. The probability of 5 of a kind can be found as we did for a royal flush: by multiplying the probabilities of 2 independent events. We find that

$$P(5 \text{ of a kind}) = \frac{13 \cdot \binom{4}{3}}{\binom{52}{3}} \cdot \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{1}{93,925},$$

making 5 of a kind 7.8 times more likely than a royal flush.

Full houses in the Card Game come in three types: $xxx|yy, xxy|xy$, and $xxy|yy$, where the | character separates the printed ticket from the community cards. The three types have respective frequencies of

$$\left[13 \cdot \binom{4}{3}\right] \cdot \left[12 \cdot \binom{4}{2}\right] = 3744,$$

$$\left[13 \cdot \binom{4}{2} \cdot 12 \cdot 4\right] \cdot \left[4^2\right] = 59,904,$$

and

$$\left[13 \cdot \binom{4}{2} \cdot 12 \cdot 4\right] \cdot \binom{4}{2} = 22,464.$$

The $xxx|yy$ hands are as numerous as full houses dealt from a standard deck.

Adding these together and dividing by $\binom{52}{3} \cdot \binom{52}{2}$ gives

$$P(\text{Full house}) \approx \frac{1}{340} \approx .0029.$$

Calculating the expected value of a Card Game ticket requires solving a linear equation in E to account for the free ticket, with expected value E, that is won by a 2-pair hand. We have

$$E = .5795 + (E+1) \cdot P(2 \text{ pairs}) - 1,$$

where .5795 is the sum of the products of the probabilities of the winning hands, 3 of a kind and higher, with their Table 1.7 payoffs. The probability of 2 pairs, .0609, is multiplied by $E + 1$ to reflect the fact that the player is not charged for the free ticket. This equation has solution $E \approx -\$0.3429$, and so the state's advantage is 34.29%—on the low side for lottery games, but very high when compared with most casino game wagers.

Variance and Standard Deviation

Expected value and house advantage do not change, relative to the amount wagered, when a player increases her bet. Another significant statistic, one that is affected by the size of the bet, is the *variance* of a random variable, which measures the deviation of the data from the expected value. A very large bet carrying a low HA might nonetheless be lucrative for a casino if the variance is high; this leads to a steady stream of income for the casino until the players run out of money. This happens more quickly, on average, in a game with high variance since more money is at risk on every decision and the casino has a greater cash reserve than any gambler.

Example 1.38. Consider 10 wagers of \$10 each at Players Choice (page 41). This gamble leads to a binomial PDF for the number of winnings X, with parameters $n = 10$ and $p = \frac{1}{6}$. The cumulative net winnings corresponding to the 11 possible values of X run from –\$100 to \$400 in steps of \$50. By contrast, wagering that \$100 on a single hand can result in only 2 outcomes: −\$100 or +\$400.

In each case, the mean of the random variable X that measures the net winnings is –\$16.67. The outcome of the single large bet, however, will be much further from the mean than the result of 10 small wagers unless the gambler either wins or loses all 10 bets. The variance of X measures this distance. ■

Definition 1.17. Let X be a random variable with expected value $\mu = E(X)$. The *variance* of X, denoted $\text{Var}(X)$ or σ^2, where σ is the Greek lowercase letter sigma, is

$$\sigma^2 = \sum_x \left[(x-\mu)^2 \cdot P(X=x)\right].$$

This represents a weighted sum of the squared distances from possible values of X to the mean μ.

The formula defining σ^2 can be rearranged to the equivalent expressions

$$\begin{aligned}
\sigma^2 &= \sum_x \left(x^2 - 2x\mu + \mu^2\right) \cdot P(X=x). \\
&= \sum_x x^2 \cdot P(X=x) - 2\mu \sum_x x \cdot P(X=x) + \mu^2, \\
&\qquad \text{since} \sum_x P(X=x) = 1. \\
&= E(X^2) - 2\mu^2 + \mu^2. \\
&= E(X^2) - \mu^2.
\end{aligned}$$

Since the variance is the sum of nonnegative terms, it is always greater than or equal to 0, and is equal to zero if and only if X can take on only one value, with probability 1.

Definition 1.18. The *standard deviation* σ of X is the square root of the variance.

As the square root of a nonnegative quantity, σ, like σ^2, is also nonnegative.

The units of variance are the units of X, squared, so if X is measured in dollars, σ^2 is measured in dollars squared. The standard deviation (abbreviated SD) measures the "typical" difference of a data point from the mean, and so has the same units as X.

Example 1.39. Continuing Example 1.38, the variance of ten \$10 bets is 3472.22 dollars squared, and its SD is \$58.93. The single \$100 bet has variance ten times greater, 34,722.22 dollars squared, and an SD of \$186.34. This SD is greater than the previous standard deviation by a factor of $\sqrt{10}$, over 3 times larger. ■

The standard deviation gives a sense of how much a typical outcome of an experiment varies from the mean. Measured in absolute dollar amounts rather than as a percentage of the wager, these SD-sized swings will bankrupt a high roller before ruining any casino that is capitalized enough to justify allowing its patrons to place such a large bet.

To a game designer, a bet with high variance can justify a low HA, provided that it can attract high rollers who will place the large bets essential to making the mathematics work out in practice. A player who loses 10 more even-money bets, of \$100,000 each, than she wins is down \$1,000,000, and while this may or may not be a financial hardship for her, a well-run casino that loses \$1,000,000 to a single player is certainly able to focus on the long term and feel little or no short-term stress. Indeed, a casino accustomed to big action like this is liable to consider a million-dollar player win as a temporary loan to that player, secure in the knowledge that, over time, the gambler will lose that money right back.

To a player, the variance is useful as a measure of the length of time necessary before the short-term performance of a wager gets "close" to the long-term expected value. A bet with high variance will eventually settle down to the long-range return, but will usually take much longer to do so than a similar bet with much lower variance. This assumes that the player does not lose all of his money before reaching that long-term average, of course, and that possibility is greater with a larger variance.

Theorem 1.10 allows us to compute the mean of a binomial random variable with a simple calculation. A similar result yields the variance and SD of a binomial RV.

Theorem 1.11. *If X is a binomial random variable with parameters n and p, then the variance of X is*

$$\sigma^2 = np(1-p) = npq$$

and the standard deviation of X is

$$\sigma = \sqrt{npq}.$$

Rake

While the twin notions of expected value and house advantage and the related quantities of variance and standard deviation are central to the operation of most casino games, including the casino carnival games with a poker flavor described in Example 1.37 and Chapter 7, live casino poker is different. Casinos and card rooms have no direct stake in the action among poker players and, unlike in other casino games, payoffs are not set in the casino's favor to ensure a profit. Since the funds at risk at a poker table simply pass among the players from hand to hand, a casino with a live poker room needs to establish a different flow of income to cover the expenses of running the room and generate a small profit. Casino poker rooms often make money by collecting an amount, called the *rake*, from each pot. The rake may be a fixed small amount from each pot, or a percentage of the action—often capped at some maximum value. These few chips are removed from the pot and dropped into a lockbox under the table, where they remain out of play.

Many poker rooms on the Las Vegas Strip assess a rake of 10% of the pot, with a maximum rake of \$4 or \$5 per hand. Some casinos with a \$4 maximum rake use special \$4 chips—which is not a standard value for circulating chips—to collect the rake. Examples of these drop chips are shown in Figure 1.6. A standard \$5 casino chip from the pot is changed into a \$4 chip and a standard \$1 chip; the \$4 chip is dropped and the \$1 chip is returned to the pot. The \$4 chips are intended for internal poker room use in collecting the rake; those few chips that leave the casino often command a premium from casino chip collectors.

It is common practice that no rake is taken at Texas hold'em tables if the hand ends before the 3-card flop is dealt; this is sometimes described as "no flop, no drop". Nonetheless, this deduction can add up: a table dealing 30 hands per hour—a reasonable pace—is pulling as much as \$150 per hour off the table and away from the players. At a high-limit game with thousands of dollars potentially in action on a single hand, this deduction may not be

FIGURE 1.6: \$4 drop chips from Harrah's Casino in Laughlin, Nevada, and the Stratosphere (now the Strat) in Las Vegas.

noticed, but at low-limit tables, the rake can be so significant that it's like having an extra player at the table who is effectively unbeatable.

The net effect of the rake is that the casino gets its money from the winning players, not from the losers. This, of course, is exactly what casinos do with other table games. At the roulette tables, winners are paid off at less than true odds, to the tune of a 5.26% house edge on most bets at an American roulette wheel. Losing players simply lose the amount they staked; winners receive less than they ought to if the bet paid out at true odds.

The rake can accumulate over time to an extent that sends most of the players' funds into the casino's drop box. If the rake is set at 10% of every pot without a maximum, it doesn't take long before this advantage consumes an entire bankroll [35].

Example 1.40. Consider a casino where the rake is 10%, at a low-limit table where the average pot size is $10, making the average rake $1 and bringing most rakes in well below any declared $4 or $5 maximum. Suppose that there are 7 players at the table and each buys into the game for $30. This $210 in cash circulates among players as they win and lose hands, except that the house rake is taken on every hand and removed from play.

If 20 hands are completed per hour, the casino collects $20 per hour. Over 6 hours of play, the casino removes $120 from the table: about 57.1% of the total buy-in. Only $90 of the original player funds remains in play; each player has effectively surrendered over half of his or her original stake to the house, with no hope of recovering it. ■

In some poker rooms, the rake has been replaced by a seat charge assessed to each player on a per-hand or per-hour basis. This is perhaps more of an equal-opportunity way for management to make money from poker, since the charge is extracted from all players, not just hand winners. The end result is the same: money traveling from the players' pockets into the casino's coffers on a one-way trip. At The Lodge Card Room in Round Rock, Texas, players must pay a membership fee for admission and are then levied an additional $11/hour seat rental fee, replacing a rake, for a seat at a table. This is at once a more consistent source of income for the card room and, ultimately, a smaller cost to the players. Membership and seat rental fees are separated from gambling funds, so players can budget their money with less concern for the effect of game play on their bankrolls.

1.8 Betting

Poker, unfortunately, is one of the few games that cannot be played so as to afford any pleasure, without the interchange of

> *money. Indeed, one might as well go on a gunning expedition with blank-cartridge, as to play Poker for "fun".*
>
> —Henry T. Winterblossom, [157].

The discussion above of the rake at a casino poker table causes us to consider more deeply the important role that betting plays in poker. Playing poker without betting is, as Henry T. Winterblossom so aptly noted in 1875, a pointless endeavor that removes almost all of the challenge and excitement from the game. Different poker games have their own conventions for placing money at risk.

Many games start out with an *ante*, where each player places some small amount of money in the central pot before the cards are dealt. This preliminary bet was devised as a defense against conservative players who could play hand after hand, folding weak hands before betting at no cost. This could happen to an extent that players with good hands had difficulty finding any action due to all the folding. In some draw poker games, this was combined with the requirement that a player hold at least a pair of jacks to open the betting; the ante was then a way to keep players in the game and to encourage continued participation after the betting had begun [157]. This rule was sometimes shortened to "Jack Pots", and is thought to have originated in Toledo, Ohio [8].

Example 1.41. Find the probability that a dealt 5-card hand—before the draw—is at least a pair of jacks.

Table 1.2 provides the necessary information. There are 198,180 hands that rank 2 pairs or better. Of the 1,098,240 one-pair hands, $\frac{4}{13}$, or 337,920, are a pair of jacks, queens, kings, or aces. The probability that a single 5-card hands is a pair of jacks or better is then

$$\frac{536,100}{2,598,960} \approx .2063,$$

so about 1 hand in 5 holds openers.

As a rough approximation to the chance that at least one player at a full table of 7 players holds openers, assume that the hands are independent. While this is not quite correct, it is correct enough for a rough approximation. The number X of players who hold at least a pair of jacks is then a binomial RV with parameters $n = 7$ and $p = .2063$. The probability that someone can open is found with the Complement Rule:

$$P(X > 0) \approx 1 - P(X = 0) = 1 - \binom{7}{0} \cdot p^0(1-p)^7 \approx .8015,$$

so someone should be able to open roughly 4 hands in every 5. ■

A philosophical objection to jacks or better is that it can force players to follow an opener and risk money even if their hands are weak, especially if the opener makes a large opening wager. This has been described as a lottery and thus contrary to the spirit of poker, which was held to be a game based on the principle of free choice for all players where their own money is concerned. Since the amount at risk could be much larger than the comparatively small required ante, the requirement of an ante was not seen to be as egregious an offense against free choice [8]. The conservative American South was hailed as an exemplar for resisting the jacks or better rule.

A second objection to jacks or better was that it revealed too much information about the opener's cards too early in the hand. Suppose that a player opens and then draws only 1 card. Since he opened the betting, his hand cannot be 4 cards to a straight or a flush, though a player might reasonably play a draw poker hand that way. By opening, a gambler has given free and important information to his opponents.

Once a player opens the hand with an initial bet after the antes, players are freed of the pair of jacks requirement, and may join in the betting regardless of their cards. A player holding 4 cards to a straight flush, for example, would not meet the minimum requirements to open but might want to stay in the game in the hope of improving her hand to a straight, a flush, or a straight flush. A player who calls the opener and then draws 1 card reveals no such information about his or her hand.

It may happen that no player holds a hand that may open. An advanced version of "jacks or better to open" raises the minimum opening hand every time no player can open. If all players must pass the opportunity to open, the cards are thrown in, everyone antes again, and in the second deal, a pair of queens or higher is necessary to open. The probability that a single hand is at least a pair of queens is approximately .1738. This escalation continues: if no player can open with queens or better, everyone antes a third time, and the players must hold at least two kings to open. If a deal is passed 4 times, often the minimum opening hand remains fixed at a pair of aces until someone can open. The final opening probability of a player holding a pair of aces or better is .1088. This practice has the potential to create large pots, which may cause players to stay in the game until the draw.

Many casino poker rooms have dispensed with the ante in favor of a pair of preliminary bets that rotate around the table: the *small blind* and *big blind*, which bear those names because they are bets made before seeing any cards. For ease of collection, two players are designated to post these bets before the cards are dealt. In successive hands, the blinds rotate around the table, so the net initial cost to a long-term player is about the same as posting an ante on every hand.

Once the initial antes are made and the first cards dealt, the first round of betting follows. Beginning with the player to the left of the dealer or designated dealer, each player acts in turn. Players may *fold*, discarding their cards and forfeiting any further interest in the pot, or they may place a bet, whose amount is sometimes constrained by set minimum and maximum values. If no player has yet made a bet, a third option is to *check*: to remain in the game without making a wager. Checking is not permitted once a wager has been made; past that point, gamblers who subsequently act may fold, may *call* the previous bet by wagering an identical amount, or may *raise* by making a larger bet. Some tables do not allow players to raise in a betting round in which they have previously checked; this is called a *check-raise* or *sandbagging*. Other tables permit check-raising.

If a player has raised, any subsequent player may call by matching the amount of the raise as well as any previous raises since their last turn to bet. Betting continues in turn until all players have either folded or placed the same amount into the pot. The number of raises may be limited by the rules of the game or by local custom.

Example 1.42. In a four-handed game, Robin is the first to act, and checks. Pat then bets \$1. Terry cannot check since a bet has been made, and so raises by betting \$3. Sandy folds. Robin raises Terry by betting \$5. Pat must bet \$4 to stay in the game, and calls by doing so. The pot currently contains \$13. Terry has 3 options: fold, bet \$2 to call and stay in the hand, or make an additional raise. ■

In a poker game such as 5-card stud or Texas hold'em, where cards are dealt one at a time separated by rounds of betting, a round where every player checks is sometimes referred to as a "free card", since players get to see the next card without risking any money. In a 1993 tournament in the French Pacific territory of New Caledonia, though, a local rule held that a round where all players check ended the hand at once, with the pot then equally divided among all remaining active players [92].

Once all bets have been equalized, the hand continues and further cards are dealt. This process repeats through every round of betting, which may number from 2 to 5 or more, depending on the game. If any cards are dealt face up, the right to make the first bet in subsequent rounds may go to the player with the highest exposed card or cards.

Bet Limits in Texas Hold'em

Texas hold'em can be broadly subdivided into two types, *limit* and *no-limit*, which differ based on the rules surrounding betting. In limit hold'em, maximum bet amounts are specified in the name of the game. For example, in a cash game of 3/6 limit, bets before the flop are restricted to \$3. Players may raise either \$3 or \$6 before the flop; the number of raises may be fixed by the rules of the poker room.

After the flop, the same \$3 limit applies, but for betting rounds after the turn and river have been dealt, the increment rises to \$6: bets must be made in multiples of \$6 and again, the number of raises may be fixed. Some poker rooms offering limit games may provide that in a two-player showdown, the number of raises is unlimited.

No-limit games have no such restrictions, and players may bet, at each opportunity, as many chips as they have in front of them. A player seeking to score a big win may choose to go *all-in* by betting all of his chips. Other players are then compelled to match the amount of this bet or fold. In poker tournaments such as the World Series of Poker, chips may not correspond to dollars; in the 2024 Main Event, a no-limit hold'em game, players paid $10,000 as an entry fee and received 60,000 chips initially.

An intermediate way of limiting bets is *pot-limit* poker. In a pot-limit game, any raise is limited to the number of chips already in the pot.

1.9 Exercises

Answers to starred exercises begin on page 335.

1.1.⋆ Unlike one-pair hands, flushes are not evenly distributed according to the highest card. Omitting straight flushes, what proportion of flushes are ace-high?

1.2.⋆ Consider a poker variation that deals 6-card hands.

a. Find the number of 6-card poker hands containing 3 pairs.

b. What proportion of 3-pair 6-card poker hands contain 3 pairs from consecutive ranks, with aces counting either high or low?

1.3. Is the number of one-pair hands in a game played with 8-card hands dealt from a standard 52-card deck greater than half the number of possible 8-card hands?

1.4.⋆ Find the probability of a 1-pair flush in the Michigan Lottery's Card Game.

1.5.⋆ Consider the Card Game ticket bearing the cards $A♣$ $K\diamondsuit$ $9♠$. Find the probability that this ticket wins a prize.

1.6.⋆ Find the expected value, prior to the draw, of a Card Game ticket bearing 3 of a kind.

1.7.⋆ In 1979, the *New York Daily News* ran a weekly sweepstakes called Payoff Poker based on straight flushes. Each week, the lowest and highest cards in a straight flush were published, for example, the $8\heartsuit$ and $Q\heartsuit$. Readers were invited to choose 3 numbers from 1–50, and at the end of the week, each remaining card in the deck was assigned a number in that range. A player choosing the 3 numbers corresponding to the 3 middle cards of the straight flush won $5000.

a. Find the probability of winning a Payoff Poker drawing.

b. There was no cost to enter Payoff Poker, except for the price of a postage stamp to mail one's entry. Given that a first-class stamp cost 15¢ in 1979, find the expected value of a single entry.

c. Calculate the variance of a single Payoff Poker entry.

Wild Cards

1.8. Give an example of a hand containing a single wild card that would be valued as a straight flush according to the Wild Card Rule.

1.9.⋆ It is customary to require poker players to designate any wild card in their hands as a card they do not already hold. A gray area here is 5 of a kind; unless a 5th suit is imagined, the wild card would necessarily duplicate a card already held.

If this restriction is not in place, a player may lay claim to a "double ace-high flush", as when holding W A♣ 7♣ 6♣ 3♣. Such a hand would beat any ace-king high flush. Find the number of possible double ace-high flushes in a deck with a single joker.

1.10.⋆ If one-eyed jacks are wild, find the number of full houses in a hand holding

a. One wild jack and one non-wild jack ($WJxyz$), where x, y, and z need not all denote different ranks.

b. One wild jack and both non-wild jacks ($WJJxy$).

1.11.⋆ In [24], we find the following statement about 5-card draw poker with deuces wild:

> Assuming that a player will average to hold a deuce when he has a natural pair to make three of a kind, low triplets are not worth much in Draw.

Find the probability that a player holding a natural pair (other than 22) also holds a deuce. The Complement Rule may be helpful. Is this a valid assumption?

1.12.⋆ In the 162-card deck with 110 wild cards described on page 35, find the probability of a 5-card hand with no wild cards.

1.13.⋆ Show that the probability of 5 of a kind in a 161-card deck with 109 wild jokers is less than 50%.

1.14. In [112], the possibility of an entire color of cards being designated wild is mentioned. If all red cards are wild, find the probability of 5 of a kind.

Chapter 2

Stud Poker

The earliest form of poker was a game called *straight poker* (page 165), which dealt each player 5 cards face down and then proceeded to a final showdown. Straight poker was in many ways a game of pure chance, since each player had to play the 5 cards initially dealt to him, with no opportunity to legally learn anything about his opponents' cards, nor any chance to improve by discarding and drawing. The game was an exercise in successive bluffing after the cards were dealt. Moreover, it was believed to be more open to cheating than later forms of poker [27]. As the game evolved, the options of exposing cards and drawing replacements were introduced and incorporated into new games.

One of the simplest poker games is *stud* poker, where players are dealt a fixed number of cards, some face up and some face down, and must build the best possible hand from those cards with no opportunity to exchange any of them. Figure 2.1 shows a 5-card stud poker hand dealt through to completion, with 1 card face down and 4 face up.

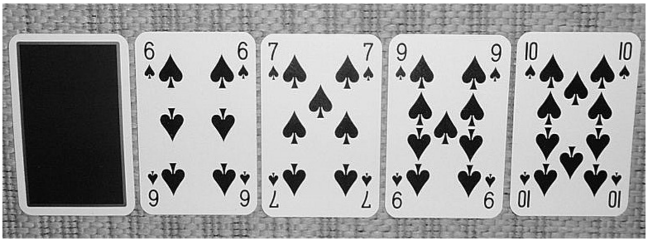

FIGURE 2.1: 5-card stud poker hand, showing 1 hole card and 4 upcards [95].

2.1 5-Card Stud

Deck composition:	52 cards
Hand size:	5 cards
Betting rounds:	4

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In *5-card stud*, the first card to each player is dealt face down. Cards dealt face down in poker games are frequently called *hole cards.* The remaining 4 cards are dealt face up, with a round of betting—and an opportunity to fold—following each card except the hole cards.

Example 2.1. Alex, Pat, and Sam face off in a game of 5-card stud. Their first two cards are shown here, with the hole cards in parentheses:

- Alex: $(T\diamondsuit)$ $6\heartsuit$.
- Pat: $(9\spadesuit)$ $5\spadesuit$.
- Sam: $(J\spadesuit)$ $3\heartsuit$.

The privilege of the first bet goes to Alex, who holds the highest open card. No player holds a particularly high or low open card relative to the others, nor does anyone have a pair, so all three check through to the third card. The new hands are

- Alex: $(T\diamondsuit)$ $6\heartsuit$ $J\clubsuit$.
- Pat: $(9\spadesuit)$ $5\spadesuit$ $5\heartsuit$.
- Sam: $(J\spadesuit)$ $3\heartsuit$ $K\spadesuit$.

With an open pair of 5s, Pat opens the betting, and is called by the others. Each player then receives a fourth card, bringing their hands to:

- Alex: $(T\diamondsuit)$ $6\heartsuit$ $J\clubsuit$ $8\diamondsuit$.
- Pat: $(9\spadesuit)$ $5\spadesuit$ $5\heartsuit$ $Q\clubsuit$.
- Sam: $(J\spadesuit)$ $3\heartsuit$ $K\spadesuit$ $6\clubsuit$.

Pat continues to hold the best open hand, and bets again. Alex and Sam see that their chance of pairing their open 6s is slim, but since they are not yet beaten and have plausible draws to hands higher than Pat's pair of 5s, they call but do not raise.

The last cards are now dealt, and the final 5-card hands are:

- Alex: $(T\diamondsuit)$ $6\heartsuit$ $J\clubsuit$ $8\diamondsuit$ $6\spadesuit$: A pair of 6s.
- Pat: $(9\spadesuit)$ $5\spadesuit$ $5\heartsuit$ $Q\clubsuit$ $9\clubsuit$: 2 pairs: 9s and 5s.
- Sam: $(J\spadesuit)$ $3\heartsuit$ $K\spadesuit$ $6\clubsuit$ $T\clubsuit$: King-high hand.

Alex successfully beat the low chance of 1 in 20 of pairing the $6\heartsuit$, and with the high hand among the exposed cards, now bets first. Pat has the best hand and raises. This is somewhat of a calculated risk, since there is a small chance that Alex now has three 6s or a higher 2-pair hand, jacks over 6s, than Pat. Since Alex has not raised prior to this round, it's a pretty good guess that that is not the case. Sam, who is clearly beaten by the cards on the table, folds. Based on the raise, Alex figures Pat for two pairs or three 5s, and so folds. ■

Gameplay Considerations

> *Stud is neither all luck nor all mathematics. It is mostly a battle of wits. In the long run the best man wins—but not necessarily the best mathematician.*
>
> —George Henry Fisher, [36].

In any poker game with cards dealt face up, an important part of playing strategy is keeping track of the cards that have been exposed, including those that have been discarded by players who have folded their hands.

Example 2.2. If you hold 3 hearts after 3 cards have been dealt in a 6-handed game of 5-card stud, then your probability of completing a flush, with no consideration of other cards, is

$$\frac{\binom{10}{2}}{\binom{49}{2}} = \frac{15}{392} \approx .0383.$$

If no other hearts are visible among the 10 exposed cards, your chance of completing a flush is

$$\frac{\binom{10}{2}}{\binom{39}{2}} = \frac{15}{247} \approx .0607,$$

just over 6%. The probability of a flush decreases with every heart on the board. ■

While few poker players are prepared to compute these exact probabilities in a live game, we can make some general statements. If there are h hearts exposed among the 10 upcards in this 6-player game, the probability of completing a flush is

$$p = \frac{\binom{10-h}{2}}{\binom{39}{2}} = \frac{(10-h)(9-h)}{1482}.$$

If we set a 3% chance of achieving a flush as a threshold, we need only solve the inequality $p \geqslant .03$, or

$$\frac{(10-h)(9-h)}{1482} \geqslant .03,$$

to see how many exposed hearts justify continuing our pursuit of a flush.

This inequality can be simplified to

$$h^2 - 19h + 45.54 \geqslant 0.$$

Recognizing that h must be an integer in the range $0 \leqslant h \leqslant 10$, we get $h \leqslant 2$: only if there are 2 or fewer hearts visible does the chance of completing a flush exceed 3%.

A strategy guideline for stud poker advises folding unless your first two cards total at least 19, with aces and face cards counted as 10 [156]. The probability of a total of 19 or higher in 2 cards is

$$\frac{\binom{24}{2} - 6}{\binom{52}{2}} = \frac{45}{221} \approx .2036,$$

a chance that may be too low for a poker player who craves action. In the numerator, the factor $\binom{24}{2}$ counts the number of ways to choose 2 cards from among the 9s through aces; we then subtract 6, the number of ways for that choice to be a pair of 9s. While a pair of 9s may be a desirable starting hand, it totals only 18 points, and so does not meet this standard.

This rule, rigidly applied, would mean folding on hands like a pair of 9s, 8s, or 7s that might be worth seeing through to another card, especially if no opponent shows another card of that rank.

Suppose, for example, that you hold a pair of 8s in your first 2 cards. What is the probability that an opponent holds a better hand?

There are 50 cards remaining in the deck, and 6 ranks that can beat a pair of 8s. Assuming for the moment that your opponents' hands are independent to simplify calculations, the probability that any one opponent can beat a pair of 8s is

$$p = \frac{6 \cdot \binom{4}{2}}{\binom{50}{2}} = \frac{36}{1225} \approx .0294.$$

If you are facing n other players, the probability that none of them can beat a pair of 8s is

$$P(n) = (1 - p)^n.$$

With 7 other players, we have $P(7) \approx .8116$.

Suppose that your final 5-card hand is a pair of 8s. What is the chance that another player has a better hand?

Of the 2,598,960 possible 5-card poker hands, 705,060 are better than a pair of 8s, making the probability of a hand that beats a pair of 8s $p \approx .2713$. The probability of n opponents all holding a 5-card hand less than a pair of 8s is, once again, approximately $(1 - p)^n$, but with the new value for p just found. Recognizing that it is unlikely that every opponent will stay in to the last card, the chance that all 7 opponents fall short of your pair of 8s is about .1041, just more than 10%.

An alternative to the 19-point strategy for the 5-card stud player is "Fold right away unless your hole card is higher than any player's upcard" [106, 156]. This strategy rests on the reasoning that the potential for a pair that beats any other player's potential pair is worth betting. If you have a king in the hole and your upcard is neither a king nor an ace, the probability that none of your n opponents holds a hidden ace is

$$\frac{\binom{46}{n}}{\binom{50}{n}} = \frac{(n-50)(n-49)(n-48)(n-47)}{5,527,200}.$$

Against 7 or fewer opponents, this probability is greater than 50%. With a hidden jack and a lower upcard, you have a 50% chance of the top upcard only if facing 1 or 2 other players. Any lower hole card is not worth backing at a large table unless you hold a hidden pair—perhaps not even then if it's a low pair.

Some Geographic Variations

Asian/Spanish Stud Poker

At Casino M8trix in Santa Clara, California, *Asian Stud Poker* is 5-card stud played with a 32-card deck: the 7s through aces in all 4 suits. As in standard stud games, the aces may be counted as low (effectively as 6s) or high when forming straights. This game is also known as *Spanish Stud*. In Maryland, this game was popular during the 19th century under the name *strip-deck poker*, which was played by groups of 4 or fewer players [37].

With 20 small cards removed, players can expect higher hands if they stay in the game, and so it is likely that pots will get bigger. The probability of a pair in the first 2 cards is

$$\frac{8 \cdot \binom{4}{2}}{\binom{32}{2}} = \frac{48}{496} \approx .0968,$$

which is a significant improvement over the probability of an opening pair from a full deck:

$$\frac{13 \cdot \binom{4}{2}}{\binom{52}{2}} = \frac{1}{17} \approx .0588.$$

However, the probability of starting with 3 suited cards falls from

$$\frac{4 \cdot \binom{13}{3}}{\binom{52}{3}} \approx .0518 \quad \text{to} \quad \frac{4 \cdot \binom{8}{3}}{\binom{32}{3}} \approx .0452.$$

With fewer cards in each suit, even 3-card flushes are harder to come by.

The depleted deck means that the hand rankings change: a flush beats a full house. The probability of a flush is

$$\frac{4\binom{8}{5} - 4(5)}{\binom{32}{5}} = \frac{204}{201,376} \approx .0010,$$

slightly more than 1 chance in 1000. The numerator in this formula is derived by multiplying the number of suits by the number of ways to choose 5 cards from the 8 in that suit, then subtracting the number of possible straight flushes. The probability of a full house is higher:

$$\frac{8\binom{4}{3} \cdot 7\binom{4}{2}}{\binom{32}{5}} = \frac{1344}{201,376} \approx .0067;$$

over 6 times greater than the chance of a flush. There are 5100 possible straights with this deck, so a straight ranks below both hands.

It is perhaps more surprising to discover that in Spanish stud, or any game played with 5-card hands dealt from this 32-card deck, a flush beats 4 of a kind. We saw that there are only 204 flushes, with straight flushes excluded. The number of 4-of-a-kind hands is $8 \cdot 28 = 224$, making them more common than flushes. Casino M8trix disregards this order, continuing to rank 4 of a kind above a flush in Asian stud poker.

Table 2.1 shows the distribution of hand types, ranked by frequency, in Asian stud poker.

Canadian Stud Poker

Canadian stud poker is a variation on 5-card stud that recognizes 2 new hands:

- A *4-flush* contains 4 suited cards and 1 unsuited card.
- A *4-straight* contains 4 cards in sequence and a fifth card not in that sequence.

TABLE 2.1: Frequencies of 5-card Asian Stud Poker hands.

Hand	Count
Royal flush	4
Straight flush	16
Flush	204
Four of a kind	224
Full house	1344
Straight	5100
Three of a kind	10,752
Two pairs	24,192
High card	52,020
Pair	107,520

Both hands beat one pair but lose to two pairs, so a 4-flush or 4-straight containing a pair will be played as the new higher-ranked hand. As with 5-card straights and flushes, a 4-flush beats a 4-straight, as the following counts will show.

There are

$$4 \cdot \binom{13}{4} \cdot 39 = 111,540$$

ways to draw 4 cards of one suit and 1 card of another suit. From this total, we must subtract the number of straights with 4 cards of one suit, which will be played as straights, not 4-flushes. These hands contain 4 suited cards within a 5-card sequence and a fifth card from one of the other 3 suits completing the straight; for example, 2♢ 3♣ 4♢ 5♢ 6♢.

These hands are easily counted. There are:

- 10 different choices for the ranks of the cards, from $A2345$ to $TJQKA$.
- 5 ways to pick which card is of the odd suit.
- 4 ways to pick the suit of the 4-flush.
- 3 ways to pick the suit of the odd card.

Multiplying gives 600 straight 4-flushes, and subtracting gives 110,940 4-flushes in total.

For 4-straights, we first separate out the hands where the lowest card in the straight is an ace or jack. Once this card is chosen, there are 4^4 ways to pick the cards in the straight and 44 ways to pick the 5th card so that it does not extend the straight to 5 cards. This gives $2 \cdot 4^4 \cdot 44 = 22,528$ 4-straights.

If the lowest card in the 4-straight is a deuce through 10, we must guard against the fifth card extending the straight at either end. There are $9 \cdot 4^4 \cdot 40 = 92,160$ of these 4-straights. Adding gives 114,688 hands with exactly 4 cards in sequence.

This number must be reduced by the number of 4-straights with 5 cards of the same suit, since they will be played as a flush. 4-straights with all 5 cards the same suit number $4 \cdot (2 \cdot 8 + 9 \cdot 7) = 316$, a count which excludes straight flushes as these were already excluded from the count of 114,688 above. This gives a total of 114,372 4-straights.

New York Stud and The Waiting Game

In *New York Stud*, only 4-straights, which are called outside straights, are recognized. An outside straight bears a pair but loses to 2 pairs.

Going the other way, Bally's Casino in Las Vegas (now the Horseshoe) briefly offered a 5-card stud variant called *The Waiting Game*, where 4-card flushes were recognized and outranked 1-pair hands. This was a \$2/\$5 limit game and was marketed as a game to play while waiting for a spot to open up at a conventional poker table.

Stud Poker With Wild Cards

5-card stud may be played with wild cards, but adding a single wild joker to the deck runs a risk of players folding quickly when the joker is revealed in an opponent's hand. Declaring all 4 deuces as wild cards slightly increases the chance of any 1 player drawing a wild card and might entice players to stay in a hand longer.

The possibility of a wild deuce in the hole is perhaps more intriguing. With 4 wild cards, the impact of an exposed wild deuce is diminished, as there are other wild cards in the deck. In an n-player game, the probability that at least 1 player has a deuce as his or her hole card is, by the Complement Rule, $1 - P(\text{No deuces in the hole})$. This probability is

$$P(n) = 1 - \frac{\binom{48}{n}}{\binom{52}{n}} = \frac{-n^4 + 202n^3 - 15,299n^2 + 514,898n}{6,497,400}.$$

$P(n)$ first exceeds 50% at $n = 9$, so in any game with fewer than 9 players, it is more likely than not that no one holds a facedown deuce. A player who catches a deuce in the hole may inadvertently reveal this fact through an aggressive betting pattern that is at odds with his or her visible cards.

Hold It and Roll It

Another option incorporating wild cards in 5-card stud poker is *Hold It and Roll It*: a game where cards are dealt face down and each player chooses, on each round, which card to turn up [159]. One card is always face down, and it and any other cards of that rank held by the player are wild for his or her hand. Once turned up, a card cannot be turned back down. The ability to choose which card to turn up at every betting round is a second chance for skill to enter into the game. As a general rule, it is better to turn up higher cards; keeping a low card in the hole opens up possibilities for higher pairs, 3 of a kinds, and higher-ranking hands.

Example 2.3. If a player holds $2299K$, concealing a deuce in the hole results in four 9s, while keeping a 9 face down results in "only" 4 deuces. Keeping the king concealed leads to a full house, which is the inferior choice for this hand and should be avoided. ■

Choosing which cards to reveal at each betting interval should, of course, prioritize pursuing as high a hand as possible. Straights and flushes are far more common in Hold It and Roll It than in standard 5-card stud.

Example 2.4. Consider a 3-player game.

- Robin is dealt $9\diamondsuit$ $7\diamondsuit$ and turns up the 9.
- Chris gets $A\heartsuit$ $K\clubsuit$ and faces the ace.
- Terry has $K\diamondsuit$ $7\spadesuit$, and chooses to expose the king.

Robin's 2-card open straight flush might be cause for some small optimism, but both other players effectively hold high pairs. These pairs are obvious to all attentive players, since we're early in the hand.

The third card is dealt to each player.

- Robin receives and immediately turns up the $T\diamondsuit$.
- Chris adds the $7\clubsuit$ and turns up the $K\clubsuit$ from the first round.
- Terry's third card is the $8\spadesuit$, and is turned up.

Robin now has an active 3-card straight flush draw. All 3 players—unbeknownst to each other—are concealing 7s, which greatly limits their chances of drawing another 7 and holding 2 wild cards of that rank.

Round 4 commences.

- Robin's fourth card is the $J\diamondsuit$, and as it's turned up, the 3-card straight flush $JT9$ of diamonds is visible.
- Chris adds the $Q\clubsuit$ and exposes it to show a 3-card straight, AKQ, with a wild 7 in the hole.
- Terry draws and turns up the $Q\spadesuit$. With an unsuited $KQ8$ showing, this hand is not looking as strong as the others.

Robin's hand will be a straight flush no matter what the fifth card is, since the $7\diamondsuit$ can be turned up and the fifth card can then be read as the $8\diamondsuit$. Chris and Terry do not know that Robin holds the $7\diamondsuit$, though, so they may well continue to bet through to the fifth card.

We move on to the final card.

- Robin receives the 4♣, which stays face down as the 7♢ is revealed. At this point, everyone knows that Robin has a straight flush: 7♢ *W* 9♢ *T*♢ *J*♢.
- Chris' fifth card is the *J*♠. By turning that up, the 7♣ can be used as a 10 to complete a straight, *A*♡ *K*♣ *Q*♣ *J*♠ *W*, which is a strong hand and would be a winner if not for Robin holding a stronger one.
- Terry draws the 8♢ that Robin was hoping for. By turning up the 7♠, Terry holds 2 wild 8s and the hand of *WWK*♢ *Q*♠ 7♠ becomes 3 kings—but Robin's straight flush wins the pot.

■

If these hands are played out as standard 5-card stud hands with no wild cards, Robin's hand drops to jack-high, Chris holds ace-high, and Terry wins with a pair of 8s.

While the guarantee of at least 1 wild card per hand means that Hold It and Roll It results in higher hands than in 5-card stud without wild cards, there are no full houses with perfect play. A player dealt a full house will hold 3 of a kind and a pair. One of those cards will be in the hole, giving either 2 or 3 wild cards that combine with the rest of the hand to make 5 of a kind.

Every Hold It and Roll It has at least 1 wild card. Hands holding a natural pair with one card of the pair staying in the hole will have 2. How many hands have exactly 2 wild cards?

With perfect play, any hand holding a natural pair can be played as a 2-wild-card hand, since the second card dealt of the pair can be—indeed, usually should be—kept face down, making it and its same-rank counterpart both wild. The number of 5-card hands with a single pair is 1,098,240 (page 22).

A dealt 2-pair hand should keep one card from the lower pair face down, giving 2 wild cards and promoting the hand to 4 of a kind. These hands number 123,552.

Dealt full houses, as noted earlier, will promote to 5 of a kind, but whether this is 3 of a kind plus 2 wild cards or a pair plus 3 wild cards depends on the ranks involved.

Example 2.5. In a hand consisting, ultimately, of three jacks and two 8s, there will always be an opportunity to play the 8s as wild and turn the hand into 5 jacks. If the last card is an 8, then it can be left face down and a jack in the hole, if there is one, can be turned up. If the 8s are dealt first: 88*JJJ*, then one will be face down, and it will remain face down as the 3 jacks are dealt and the hand advances from 3 jacks (rather than 3 8s) to 4 to 5.

If, on another hand, the card order is *JJJ*88, the first jack is left face down until the first 8 appears at card #4. Facing the jack and concealing the 8 gives 4 jacks, which beats the four-8 hand resulting from keeping a jack in the hole. When the second 8 appears, the hand completes to 5 jacks.

There are

$$\frac{5!}{3! \cdot 2!} = 10$$

distinguishable permutations of three jacks and two 8s. In any of the other eight orders, there will never be a time when the player is forced to declare the jacks wild.

If the hand is instead three 8s and two jacks, the goal will be to keep an 8 in the hole so that all 3 can be wild, and again, this is always possible with smart play. ■

If we assume that the skilled player will seek to turn their hand into the highest 5 of a kind possible, then half of all full houses, a total of 1872 hands, will be played as 3 of a kind plus a wild pair. These are the hands where the 3 of a kind outranks the pair.

We find that the probability of a hand with 2 wild cards is

$$p = \frac{1,098,240 + 123,552 + 1872}{2,598,960} \approx .4708$$

—nearly 50%. As a hand with 2 wild cards will rank no lower than 3 of a kind, we see that much higher hands are required to win in Hold It and Roll It.

2.2 7-Card Stud

Deck composition:	52 cards
Hand size:	7 cards
Betting rounds:	5

In a game of 5-card stud among skilled players, many players will fold their hands early, when their first 2 cards show little promise for improvement, either on their own or in the light of a strong exposed card in an opponent's hand. The 19-point minimum strategy described on page 56 may be good for a player seeking to play a sound game, but it's not conducive to many-handed poker. Moreover, the size of the pots can be quite small when few players persist past the first 2 or 3 cards.

Some modifications to 5-card stud have the effect, intended or not, of increasing the number of gamblers who stay with a hand until its conclusion. One of these is *7-card stud*, where players are dealt 7 cards and must make their best 5-card hand out of their holdings. In 7-card stud, each player is initially dealt 3 cards, two face down and one face up, before the first round of betting. The fourth, fifth, and sixth cards are dealt face up, and each is followed by another round of betting. Finally, each remaining player's seventh card is dealt face down before the fifth and final betting round.

Given 7 cards from which to make a 5-card hand, there are

$$\binom{7}{5} = 21$$

possible 5-card hands. The possibility of redeeming a poor starting hand with 2 extra cards to choose from may well encourage some players to call some early bets in order to see more cards.

Since the basic hand types described in Chapter 1 are unchanged in 7-card stud, we may be interested in the distribution of final 5-card hands in 7-card stud. There are, of course, $\binom{52}{7} = 133,784,560$ ways to deal a 7-card poker hand, but since players are required to choose their best 5-card hand from 7 cards, this number is less important than $\binom{52}{5} = 2,598,960$, the number of 5-card hands that we've been working with.

In the case of 4-of-a-kinds, there are

$$13 \cdot \binom{48}{3} = 224,848$$

7-card hands containing four of a kind. These hands are easily counted since a hand with 4 cards of the same rank cannot take on 3 cards and "promote" to a higher-ranked straight flush or royal flush.

The frequencies of the various hands are listed in Table 2.2.

TABLE 2.2: Frequencies of 5-card poker hands in 7-card stud [72].

Hand	**Count**
Royal flush	4324
Straight flush	37,260
Four of a kind	224,848
Full house	3,473,184
Flush	4,047,644
Straight	6,180,020
Three of a kind	6,461,620
Two pairs	31,433,400
Pair	58,627,800
High card	23,294,460

Example 2.6. Consider the probability of a straight or royal flush in 7 cards. We can count these by focusing on the highest card in the straight. The number

of royal flushes (ace-high) is simply

$$4 \cdot \binom{47}{2} = 4324,$$

since there are 4 ways to pick the suit of the royal flush and then 47 cards remaining from which to choose the last 2.

For straight flushes with a king through 5 as the highest card, we must remove 1 card from the pool from which the 2 non-matching cards are chosen: the card that would promote the straight flush to a higher straight flush. If the straight flush is 76543♣, for example, then including the 8♣ as an odd card would throw the hand into the set of 8-high straight flushes.

It follows that there are

$$4 \cdot \binom{46}{2} = 4140$$

straight flushes headed by a 5 through king. Adding up gives $9 \cdot 4140 + 4324 = 41,584$ 7-card hands that include a 5-card straight or royal flush. Dividing by $\binom{52}{7}$ gives the straight flush probability shown in Table 2.2. ■

As we saw in Chapter 1, adding more cards to a hand improves the likelihood of making a pair or better. In 7-card stud, the median hand using traditional hand rankings is a pair, and high-card hands are scarcer than both one-pair and two-pair hands.

Example 2.7. If your first 3 cards in a hand of 7-card stud are all the same rank, find the probability of drawing out to a full house.

You hold the hard part of a full house: 3 of a kind. To get a full house, you need to draw a pair among your last 4 cards, *without* drawing the fourth card of the rank you hold or 4 cards of a different rank, because either draw would give you 4 of a kind, not a full house. We have

$$P(\text{Full house} \mid 3 \text{ of a kind}) = \frac{12 \cdot \binom{4}{2} \cdot \binom{44}{2} - \binom{12}{2} \cdot \binom{4}{2}^2 - 12}{\binom{49}{4}} \approx .3102.$$

- The factor $12 \cdot \binom{4}{2} \cdot \binom{44}{2}$ counts the number of ways to draw a pair and 2 other cards that are neither the same rank as the pair or the 3 of a kind already held.
- We subtract $\binom{12}{2} \cdot \binom{4}{2}^2$, which counts the number of ways to be dealt 2 pairs in 4 cards. These 4-card combinations were counted twice in the first factor, and so we remove one copy of each.

- Finally, we subtract 12, the number of ways to draw 4 of a kind in the last 4 cards.
- The denominator counts the number of ways to fill in the last 4 cards from the 49 unknown cards.

■

Advanced mathematics is not necessary to appreciate that the best start to a 7-card stud hand is 3 aces [71]. The probability of being dealt 3 aces off the top of the deck is

$$\frac{\binom{4}{3}}{\binom{52}{3}} = \frac{1}{5525}.$$

While this is a rare hand, it raises a question: What is the likelihood of improving a 3-ace starting hand?

The easiest calculation is the likelihood of catching the 4th ace, or of drawing 4 cards of another rank, and improving to 4 of a kind. This has probability

$$\frac{\binom{48}{3} + 12}{\binom{49}{4}} = \frac{17,308}{211,876} \approx .0817.$$

The probability of any 4 of a kind at the start of the hand is approximately .0017, so this represents a big improvement.

There are 2 ways to turn AAA into a straight flush or royal flush: the remaining 4 cards must be either $TJQK$ or 2345 in the suit of one of the aces. This probability is

$$\frac{6}{\binom{49}{4}} \approx 2.832 \times 10^{-5}.$$

For AAA to improve to a straight, the final 4 cards must be either 2345 or $TJQK$, and they must not all be of the same suit as one of the aces. There are $2 \cdot (4^4 - 4 + 1) = 506$ ways to draw out to a straight that is not a flush. The corresponding probability of a straight beginning with AAA is

$$\frac{506}{\binom{49}{4}} = \frac{11}{4606} \approx .0024.$$

Promoting AAA to a flush requires that the final 4 cards all be of the same suit as one of the aces, except for the two combinations that would yield a straight flush. These cards can be drawn in

$$3 \cdot \left[\binom{12}{4} - 2\right] = 1479$$

ways, making the probability of a flush approximately .0070.

To draw 4 cards to AAA and get a full house requires one of the following 3 draws. In each case, none of the 4 additional cards can be the fourth ace.

- *4 cards of the form* ***xxyz****, where* $x \neq y \neq z$. This is the most common path to a full house starting with 3 aces. There are

  $$12 \cdot \binom{4}{2} \cdot \binom{11}{2} \cdot \binom{4}{1}^2 = 63,360$$

  ways to draw 4 cards in this configuration.

- *Two pairs of different ranks, as* ***xxyy*** *with* $x \neq y$. This hand can be drawn in

  $$\binom{12}{2} \cdot \binom{4}{2}^2 = 2376$$

  ways.

  If $x = y$, then this hand becomes four of a kind plus an ace, not a full house.

- *Three of a kind and a fourth non-matching card:* ***xxxy*** *where* $x \neq y$. Such hands will be read as aces full of xs, since x must be a rank lower than ace. There are

  $$12 \cdot \binom{4}{3} \cdot 44 = 2112$$

  such hands.

Adding everything up gives

$$P(\text{Full house} \,|\, 3 \text{ initial aces}) = \frac{67,848}{\binom{49}{4}} \approx .3202.$$

The chance of improving an initial 3-ace hand is therefore approximately

$$.0817 + 2.382 \times 10^{-5} + .0024 + .0070 + .3202 \approx .4113,$$

just over 41%.

Gameplay Considerations

One piece of advice offered to 7-card stud players is "If your hand isn't completed in the first 5 cards, fold. And don't go for inside straights" [98]. An *inside straight* is 4 cards toward a straight with a missing rank in the middle, such as 3467, which can only be filled out to a straight by 1 rank.

This sounds like good advice, but is it overly conservative? Incomplete 5-card hands that are worth pursuing typically include 4 cards to either a straight or a flush. Suppose your first 5 cards in a 7-handed game are an open-ended 4-card straight, such as 5♠ 6♣ 7♣ 8♠, together with an unhelpful fifth card, the $Q\diamondsuit$. What is your probability of completing the straight?

There are 17 cards remaining in the deck. You need a 4 or a 9 in your last 2 cards to form a straight; the probability of getting at least 1 depends on how many of those cards are already out, in your opponents' 30 upcards and hole cards. If we denote that number by x, then the probability of catching either 1 or 2 of the cards you need in your next 2 cards is

$$\frac{(8-x)\cdot(9+x)+\binom{8-x}{2}}{\binom{17}{2}} = \frac{x^2-17x+200}{272}.$$

If $x = 0$, so 8 of the 17 cards left to be dealt will complete your straight, you have a 73.5% chance of drawing into a straight, but it is unlikely that $x = 0$. The probability that 26 cards (including your 5 cards along with your opponents' upcards) dealt from a standard deck contain no 4s or 9s is

$$\frac{\binom{44}{26}}{\binom{52}{26}} \approx .0021,$$

less that ¼%. Moreover, your hand's potential must be weighed against the possibility that an opponent will hold a higher hand. The median winning hand in a 7-handed 7-card stud game is a 7-high straight [72], so you are looking at a completed straight that scarcely exceeds the median, making it quite possible that an opponent holds a better hand at the showdown.

Inside straights are an even bigger gamble, since only one rank will fill out the straight. Consider the situation above, but assume that you hold the 4♣ instead of the 7♣, so the only way to fill in the straight is to draw a 7 in the 2 remaining cards. The best case is where none of your opponents has a 7 face up; your chance of filling the gap is then

$$1-\frac{\binom{13}{2}}{\binom{17}{2}} \approx .4265$$

by the Complement Rule. A bet that fails over half the time—especially against 6 opponents—is probably not worth making, and this is the best possible scenario.

While there may be a few very rare exceptions, this advice is pretty sound.

Variations

Chicago

Chicago is a variation of 7-card stud where deuces are wild and the pot is divided at the showdown. The division is not 50% each to the high hand and the low hand, but instead awards half the pot to the highest hand and the other half to the player with the highest spade in the hole. This rule change is cause for different player strategy: an ace-high hand with the $A♠$ in the hole is guaranteed to win half the pot, and so a player with that hand will stay in until the end, perhaps even raising in an effort to build the pot and increase his eventual 50%. A player with a good but not great hand may decide to call all bets in the hopes of securing a high spade on the 7th card and so qualifying for half the accumulated money.

Example 2.8. Suppose you are in a 6-handed game of Chicago and hold the $K♠$ as one of your hole cards. What is the probability that none of your opponents holds the $A♠$ in the hole at the showdown?

Assuming, of course, that the $A♠$ is not visible in anyone's hand, there are 15 opponent hole cards to consider, 27 exposed cards including the cards in your hand, and 25 unknown cards. The probability that none of the 15 hole cards is the $A♠$ is

$$\frac{\binom{24}{15}}{\binom{25}{15}} = .4000.$$

The chance that you hold the high spade in the hole is good, but not overwhelmingly so. ■

If there are no spades among the players' hole cards, the entire pot is awarded to the high hand. In a 4-player game where all players remain in the hand until the showdown, what is the probability that no one has a spade in the hole?

With 4 players, there are 12 hole cards. We need not consider who holds which cards; we need only look at the hole cards collectively as a set of 12. The probability that none of these cards are spades is

$$\frac{\binom{39}{12}}{\binom{52}{12}} \approx .0189,$$

close to 1 time in 53 hands.

8 Ball

The format of *8 Ball* is that of 7-card stud, with 2 cards dealt face down, 4 face up, and a 7th and final card face down. What changes is how player hands are ranked. All 7 cards may be used in building an 8 Ball hand, and hands are ranked by the number of 8s, or 2-card combinations adding up to 8, that they contain [159]. The highest possible 8 Ball hand is five 8s, either 4 natural 8s and 1 two-card combination or 3 natural 8s and 2 combinations. The requirement for 3 or 4 natural 8s means that there can only be 1 hand of five 8s in a single round of 8 Ball—here is another poker game with an unbeatable hand.

The number of hands with five 8s is found by dividing them into two types:

1. *Hands with 4 natural 8s and 1 pair summing to 8.* These hands have the form $8888x(8-x)y$ where $1 \leqslant x \leqslant 7$, and may be further subdivided according to whether or not $x = y$. The case where $x = 4$, so the hand is $888844y$, requires special handling.

   (a) The hands where $x \neq y$ number

   $$3 \cdot 4 \cdot 4 \cdot 40 + \binom{4}{2} \cdot 40 = 2310,$$

   where the first term in the sum counts hands with no paired 4s and the second counts hands with a pair of 4s.

   In $3 \cdot 4 \cdot 4 \cdot 40$, the 3 counts the number of ways to choose a pair of cards—A/7, 2/6, or 3/5—adding up to 8 and the two factors of 4 are the number of ways to pick the suit of each card. There are then 40 ways to pick the seventh, nonmatching, card.

   (b) There are

   $$6 \cdot \binom{4}{2} \cdot 4 + \binom{4}{3} = 148$$

   hands of the form $8888x(8-x)x$.

   Adding gives 2458 hands with 5 8s including 4 natural 8s.

2. *Hands with 3 natural 8s and 2 pairs summing to 8.* These hands look like $888x(8-x)y(8-y)$, which includes the case where $x = y$ and the hand consists of 3 of a kind (the 8s) and 2 pairs or 3 of a kind and four 4s. There are

   $$\binom{4}{3} \cdot \left[3 \cdot 4^2 \cdot 4^2 + 1 + \binom{6}{2} \cdot 4^2 \cdot 4^2 + \binom{4}{2} \cdot 3 \cdot 16 \right] = 19,588$$

   of these hands.

The probability of a hand with five 8s in it is then

$$P(5\ 8\text{s}) = \frac{2458 + 19,588}{\binom{52}{7}} \approx \frac{1}{6068}.$$

6-Card Stud

Deck composition:	52 cards
Hand size:	6 cards
Betting rounds:	5

We've seen 5-card and 7-card stud. What about 6-card stud?

It's been tried, but there's no uniform agreement on how the cards should be dealt. One version starts with 2 hole cards and one card face up before the first betting round, with the remaining 3 cards dealt face up and separated by betting. Another begins with 2 cards, one face up and one face down, and closes with the sixth card dealt face down [82]. In each case, a completed hand has 2 hole cards and 4 exposed cards. Either way, 6-card stud offers less potential to draw into a high hand than 7-card stud, and has less tradition behind it than 5-card stud. Taken together, this means that the game is likely to be no more than a footnote in the wide array of poker games.

2.3 Razz

Deck composition:	52 cards
Hand size:	7 cards
Betting rounds:	5

Six-card stud, played for lowball, bears a historical footnote as an ancestor of *razz*, which is simply 7–card stud played for lowball, where straights and flushes do not count as high hands and aces always rank low. Should the showdown involve a pair of aces and a pair of 2s, the aces rank lower, but both hands lose out to any high-card hand, even $KQJT9$.

With 7 cards to choose 5 from, a pair or high card does not take a hand immediately out of the running for the pot. Nonetheless, some players at a full (8-player) table prefer to fold any 3-card starting hand which is not 8-high or better. The probability of drawing 3 cards that are all 8 or lower (including aces, which are low in razz) without a pair is

$$\frac{\binom{8}{3} \cdot 4^3}{\binom{52}{3}} = \frac{3584}{22,100} \approx .1622,$$

just less than 1 chance in 6.

On a practical level, this advice may be sound, but the player who craves action may find it too limiting. Thinking more broadly, the probability of 3 cards of different ranks with the highest card no bigger than x is

$$\frac{\binom{x}{3} \cdot 4^3}{\binom{52}{3}} = \frac{8x(x-1)(x-2)}{16,575},$$

which is less than 50% unless $x \geqslant 12$. A player seeking to play past the first 3 cards more than half the time will have to bet some 3-card hands that are queen-high or higher.

Example 2.9. The best razz hand is a wheel: $5432A$ of any suits. The probability that the first 3 cards contain 3 cards of a wheel is

$$\frac{\binom{5}{3} \cdot 4^3}{\binom{52}{3}} \approx .0290.$$

The probability of drawing a wheel in 7 cards can be computed by considering five subcases. A wheel takes the form $A2345xy$ in some order.

1. x and y are different high cards (higher than a 5). There are

$$4^5 \cdot \binom{8}{2} \cdot 4^2 = 458,752$$

ways to make a wheel of this form.

2. $x = y$ to form a high pair. This forms a wheel in

$$4^5 \cdot 8 \cdot \binom{4}{2} = 49,152$$

ways.

3. x and y are the same low rank, forming 3 of a kind in addition to a wheel. These hands number

$$5 \cdot \binom{4}{3} \cdot 4^4 = 5120.$$

4. x and y contain one low rank, forming a pair, and one high rank. This leads to

$$5 \cdot \binom{4}{2} \cdot 8 \cdot 4^5 = 245,760$$

7-card wheels.

5. $x \neq y$, but both are low cards, so the hand contains 2 pairs and a wheel. This may be dealt in

$$\binom{5}{2} \cdot \binom{4}{2}^2 \cdot 4^3 = 23,040$$

ways.

Adding up cases 1–5 gives 781,824 wheels in 7-cards. The probability of a wheel is

$$\frac{781,824}{\binom{52}{7}} \approx .0058,$$

approximately 1 chance in 171. ■

Successful razz players pay close attention to their opponents' upcards and should account for the possibility that a competitor's exposed low card might be paired among their hole cards and thus considerably less valuable. If you see $J♣\ 2♢\ A♡$ in a player's third through fifth cards, the chance that either the ace or deuce is paired in the hole is

$$1 - \frac{\binom{43}{2}}{\binom{49}{2}} = \frac{13}{56} \approx .2321,$$

or close to 25%. It should be noted that this does not take any other known cards into consideration, including the 5 cards in your own hand.

One difference in the roles of exposed cards in razz vs. 7-card stud is that it is less dismaying to see your hole card duplicated among an opponent's upcards. If, for example, you hold the 5♠ in the hole, every exposed 5—instead of reducing your hand's potential to improve—decreases the chance that you will pair your hidden 5 and diminish its value. While your opponent has a start on a good low hand, you remain in a competitive position.

2.4 Exercises

Answers to starred exercises begin on page 335.

5-Card Stud

2.1.* In a game of 5-card stud, you hold $(K)QJ$, with the K in the hole and no face cards or aces visible among your 4 opponents' cards. Since your hole card beats everything you can see, you continue to bet. Find the probability that you will be dealt at least 1 face card among your next 2 cards.

2.2.* Find the probability that, in a head-to-head showdown at 5-card stud, two players tie with a pair of aces apiece.

2.3.* Compare the probabilities that your first 3 cards are in sequence in standard 5-card stud and Spanish Stud (page 57). Assume that the ranks are consecutive (so 3-card hands such as $9JQ$ are not counted here, though they have straight potential) and that $A78$ is a valid sequence in Spanish Stud.

2.4. Suppose that a variant of Canadian stud poker also recognized the 4-straight flush, with 4 cards suited and in sequence, as a separate hand.

a. How many 4-straight flushes are possible?

b. Where should this hand rank?

2.5.* How many Hold It and Roll It hands contain exactly 1 wild card?

7-Card Stud

2.6.* Calculate the number of 5-card full houses in 7-card stud that are aces full and kings full. What accounts for the difference between these two values?

2.7.* On page 5, we noted that any 5-card straight must contain either a 5 or a 10. In 7-Card Stud, where players construct their best 5-card hand from 7 dealt cards, are there more straights containing a 5 or containing a 10?

2.8.* Three of a kind in the first 3 cards of a 7-card stud hand is called *rolled-up trips*, and is especially valuable because the hand's strength is concealed. What is the probability of rolled-up trips?

2.9.* A *premium* starting hand at 7-card stud is any 3-card hand equal to or better than a pair of 10s, including three of a kind. Tens are included as a premium holding because a player holding two 10s can block many opponent straights. Find the probability of a premium hand.

2.10.* A *skip straight* in 7-card stud is a 3-card starting hand in which the cards are spaced exactly 1 rank apart; 468, for example. Find the probability that a player is dealt a skip straight.

2.11.* Find the probability that a 7-card stud hand that starts with 3 aces does not improve and winds up as 3 of a kind.

2.12. Suppose that your first 3 cards in a hand of 7-card stud are $A\diamondsuit\ A\clubsuit\ 2\heartsuit$. Find the probability that your final 7-card hand is 2 pairs: aces and deuces. (Note that with 4 cards yet to be drawn, this hand has both straight and flush possibilities.)

2.13. Your first 3 cards in a hand of 7-card stud contain a pair in the hole and an unmatched upcard. If your final hand is 2 pairs, is it more likely that you paired your upcard or drew a pair of another rank?

2.14. Consider an 8-player 7-card stud game. After the first 3 cards have been dealt, you hold 3 clubs and are interested in your chances of drawing into a flush. At this point, you know 10 cards: 7 opponent upcards plus your 3-card hand.

a. Find the probability $P(x)$ of completing the flush as a function of the number of exposed clubs, 0–7, in your opponents' hands.

b. If your threshold for pursuing a flush is that the chance of completing it is 1 in 7 (14.29%), what is the maximum number of visible clubs before you abandon that quest?

It is of interest to note that if every player in this game stays through to the 7th card, the deck will be exhausted before the dealer can complete the deal. If this happens, common practice is to designate the final dealt card (card #49) as a community card that counts as the 7th card in everyone's hand. It is not necessary to consider this possibility in solving this problem.

Razz

2.15.* The best 3-card starting hand in razz is 32A, suited or unsuited. Find the probability of starting a razz hand with 32A.

2.16.* The first bet in a hand of razz is made by the player with the worst, or highest-ranked, upcard. In the event that two or more players have equal-ranked cards, the tie is broken by looking at suits in alphabetical order. Clubs rank lowest, followed successively by diamonds, hearts, and spades, so that the K♠ is the highest card.

In a 7-handed game of razz, find the probability that two or more players have a king as their upcard, necessitating the use of suits to identify the first bettor.

Other Stud Games

2.17.* In a 4-player game of Chicago with all 4 players staying in until the showdown, find the probability that the high spade in the hole is the 2♠.

Four-Card Stud

Deck composition:	52 cards
Hand size:	4 cards
Betting rounds:	3

2.18. *4-card stud* is a variation on stud poker where players are challenged to make the best 3-card hand from 4 cards. The four cards are dealt in the following order:

- One card face down and 1 face up, followed by a betting interval.
- A third card dealt face up, also followed by a round of bets.
- The fourth and final card dealt face up, with the last round of betting leading into the showdown.

Three-card poker hands are ranked in the order shown in Table 2.3.

TABLE 2.3: Ranking of 3-card poker hands.

Straight flush
Three of a kind
Straight
Flush
Pair
High card

Confirm that a 3-card straight beats a 3-card flush by counting the number of both hands.

Chapter 3

Draw Poker

There is far less trouble caused by a Draw Poker game, properly conducted, than any other form of gambling.

—F.R. Ritter, [123].

Stud poker leaves the gambler at the mercy of the deal: with no opportunity to exchange cards, the not-inconsiderable place of skill in stud poker is restricted to betting, bluffing, and knowledge of card mathematics. *Draw poker* allows players the opportunity to discard some cards from their initial hand and receive replacements, which increases the role of skill in the game. Prior to the rise of Texas hold'em, many people's first experience with poker came in informal private games of 5-card draw poker.

3.1 Introduction

Deck composition:	52 or 53 cards
Hand size:	5 cards
Betting rounds:	2

A traditional hand of draw poker begins with 5 cards dealt face down to each player, followed by a round of betting. Players who don't fold their initial hands are then offered the chance to discard any cards they don't want and draw new cards to replace them. A player holding a pair will likely discard 3 cards, hoping to improve to 3 of a kind or higher, although she may choose to draw only 2 cards to give the impression that she holds a stronger hand. Once all hands are restored to 5 cards, a second round of betting follows. At the showdown, the player still in the game with the highest-ranked hand wins the pot.

If the game is played under the "jacks or better to open" rule, a player who starts the betting may be compelled to place her discards aside rather than returning them to the dealer, to show if requested that she initially held openers. This could happen, for example, if the player held $Q♠$ $Q♣$ 8♣ 5♣ 4♣ and opted to discard the $Q♠$ and draw to her four-flush. A player who cannot

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prove, on demand, that she originally held openers will ordinarily have her hand declared dead, or out of play. Local customs may modify this rule.

The history of poker records a number of local customs which have been created to address the split openers question [25]. One method called for the player splitting his openers to "nail" his discards to the table by placing them under a chip so that they could be inspected by any interested opponent at the hand's end. There were several objections to this rule:

- A player holding a pair of aces could nail cards without splitting openers. That could mislead opponents about his holdings.
- This was countered by the requirement that a splitter announce the split out loud when nailing the discards. Properly followed, though, this was a giveaway of information that put the opener at what was perceived to be a meaningful disadvantage. Since there was no compensating advantage that accrued to the opener, this was thought to be unfair.

A different proposed rule change allowed players to open on a 4-flush or 4-straight which held no extra value at the hand's end. This meant that a player could reasonably discard 1 card, eliminating the need to nail discards. Mathematically, this amounted to taking the 594,564 hands of jacks or better and adding to them 573,916 hands that contain 4 cards to a flush or straight without containing at least a pair of jacks [25]. Considered as groups of hands, these two groups are close to the same size, which gives credence to the argument that they should be treated the same way for the purpose of opening a hand.

This change raised the probability that a player held an opening hand to

$$\frac{1,168,480}{2,598,960} \approx .4496,$$

making it more likely that any one player could open the hand.

As noted above, one of the most likely reasons for a player to split openers is to break a high pair in order to draw to a 4-card flush or straight. Another proposed rule addressed this case by requiring a player wishing to break a pair of jacks or better to set both cards of the pair to the side when drawing and to receive 1 card. If the drawn card completed a straight or flush, the player opening could retrieve the necessary card. If, on the other hand, the player drew into 3 of a kind, he could not take back both openers. This essentially forced the player to keep track of two hands simultaneously, and may have been a solution that was worse than the problem it was purported to address.

In the early 20th century, the New York *Sun* conducted a poll of its readers in an effort to determine the "best" way to resolve the split openers question. Seven options, including several of those listed above, were proposed; readers overwhelming opted for the seventh on the list [25]:

> The opener of the jackpot must always place his discard under the chips in the pool, no matter what he has or what he is drawing

> to. He is then at liberty to split as he likes, or to disguise his hand in any manner he sees fit, and is under no obligation to say whether he is splitting or not. If there is any dispute as to his opening qualification, the discard is there under the chips in the pot to show what he had.

This rule treats all discards equally, whether the opener is splitting or preserving his openers, and so addresses the concern that an opener might be forced to reveal too much information. At the same time, the question of whether a player originally held openers is easily resolved, since the discards are nailed on every hand.

Example 3.1. Consider the following assertion from a 1915 book on poker.

> *An opener draws 3 cards and shows three 8s. If the fourth 8 is not among his discard, his hand is foul.*
>
> Algernon Crofton and R.F. Foster, [25].

This may require some explanation. If the fourth 8 was discarded, then the player opened with four of a kind, which is better than a pair of jacks, and split the openers—why a gambler would do this is a different question. It may be that he was seeking to mislead his opponents about the strength of his hand, or perhaps he felt that he was in a crooked game and wanted to throw the cheating opponents off. A dishonest dealer with a reasonable level of skill could manipulate the draw whether a player dealt 4 of a kind drew zero cards or one card, but the deck would be disrupted if the player asked for 2 cards.

If the fourth 8 was not discarded, then the player must have opened with a pair of 8s, which is not high enough to open.

If the player drew three 8s, then presumably the other two cards are a pair of jacks or better, and the hand is a full house and not merely three of a kind. If not, then he opened on less than a pair of jacks, and so his hand is foul. He forfeits any interest in the pot as well as any money he wagered, even if three 8s is the best hand. ■

3.2 Gameplay Considerations

> *[T]here is always a well-known solution to every human problem—neat, plausible, and wrong.*
>
> —H.L. Mencken, 1920.

This quotation from Mencken's essay "The Divine Afflatus" is readily applicable to certain draw poker questions that crop up during the play of a hand. Sometimes the most obvious way to play a hand turns out to be less advantageous mathematically.

A challenge to the mathematical analysis of draw poker is the distinction between a hand's absolute expected value, which is easily calculated by computing its probability, as if in a vacuum with no other players at the table, and what may be called its relative expectation, which takes into account a hand's rank compared to other players in the game [8]. A hand of three 7s may rank in the top 2% of all poker hands (page 22), but if an opponent holds three 9s, that hand has no more value than a lowly high-card hand.

Example 3.2. When drawing one card to a 4-card straight, there are 2 possibilities to consider.

- An *outside straight* is four cards comprising a straight that is open at both ends, such as 6789. Drawing either a 5 or a 10 will complete this straight. This is also called a *bobtail straight*.
- An *inside straight* is four cards comprising a straight, but with a "hole" in the middle, such as 4568 or 5689. Only a drawn 7 will complete either of these straights, so an inside straight has less potential for improvement, and thus is less valuable, than an outside straight. A 4-card straight containing an ace, such as $A234$, also qualifies as an inside straight, since only one rank will fill it out to a straight.

The probability of completing an outside straight with a 1-card draw is $\frac{8}{47} \approx .1702$, or just over 1 chance in 6. Completing an inside straight is half as likely; the probability is $\frac{4}{47} \approx .0851$. The common advice "Never draw to an inside straight" is sound, since only about 1 hand out of every 12 will draw a card of the missing rank that completes the straight. ■

In a 4-player game, suppose that you hold 7♡ 7♣ 3♠ 3♣ 9♠ and discard the 9, hoping to draw into a full house. A simple assessment of the chance of getting one of the remaining 7s or 3s would suggest that the probability is $\frac{4}{47}$, since there are 47 unknown cards and 4 of them will complete your full house.

An observer might reasonably object to this simple analysis, noting that if your opponents hold any of the 7s or 3s that you need, the 4 in the numerator above is too large. Another observer, on hearing this argument, might correctly point out that if none of your opponents hold a card that you need, the denominator is too large, and thus $\frac{4}{47}$ *underestimates* your probability of success. We can compute the probability of a full house taking the other hands into account.

The probability that the 15 cards your opponents hold contain k 7s and 3s, where $0 \leqslant k \leqslant 4$, is

$$P(k) = \frac{\binom{4}{k} \cdot \binom{43}{15-k}}{\binom{47}{15}}.$$

Since the distribution of the cards among hands does not matter, we simply consider your opponents' cards as a set of 15. This function has a maximum value of approximately .4171, when $k = 1$.

The probability that you complete a full house if the 15 cards in their hands contain k 7s or 3s is then

$$\frac{4-k}{32},$$

since if we know how many of your desired cards are in other players' hands, the number of unknown cards drops from 47 to 32.

Summing across all possibilities gives

$$P(\text{Full house}) = \sum_{k=0}^{4} \left(P(k) \cdot \frac{4-k}{32} \right) = \frac{4}{47}.$$

We have the same result either way. This justifies the use of the shorter formula in assessing the likelihood of improving this hand.

Example 3.3. However, there are times when the rest of the deck cannot be ignored. Consider the hand 10♡ 10♢ A♢ J♠ 6♢. If we seek the probability of drawing into a full house when discarding 3 cards and holding the pair of 10s, the calculations here are complicated by the discards. It is far less likely that the hand will be completed to 3 aces and a pair of 10s than, say, to three kings and two 10s, since the A♢ is no longer available in the draw. ■

Using the short calculation, we can confirm that drawing 1 card and converting an inside straight to a straight has the same probability as improving a 2-pair hand to a full house with a 1-card draw. In either case, there are 4 cards that complete the hand as indicated, making the probability $\frac{4}{47} \approx .0851$.

Drawing to an inside straight is not recommended because the starting hand is weak. If the draw fails to complete a straight, the best that the hand can be is one pair. The better course of action is to fold unless the pot is very large.

How large is "very large"? An exception to "never draw to an inside straight" takes the size of the pot into account. If the pot is sufficiently large, it may be worth drawing to a longshot. The odds against filling in an inside straight are 11–1, so if the pot is at least 11 times what it would cost to call all bets and advance to the draw, making the bet and drawing 1 card is favored.

This is an example of *pot odds*. This assumes that a straight is a sure winner if the draw is successful, which is an acceptable first approximation whose accuracy decreases as more players stay in the hand to the draw.

For a 1-card draw to an outside straight with 8 cards to complete it, the chance of winning doubles to $\frac{8}{47}$, about 1 in 6, and so the pot should be 5 times more than the necessary bet to make the draw worth taking.

We have shown that drawing 1 card to 2 pairs and completing a full house has the same probability as filling an inside straight with a 1-card draw. A player might reasonably think that drawing a single card is the only way to play this relatively strong hand, which ranks in the upper 10% of all 5-card hands.

Mencken's maxim above suggests a deeper look. If the hand is $AAKKx$, then the player holds the top 2-pair hand, and should rightly draw against long odds in the hope of completing a full house. However, when holding two low pairs, such as $4433x$, it's worth considering the alternative of discarding 3 cards and trying to improve the remaining pair to 3 of a kind or better—since $4433x$ will lose to almost any other 2-pair hand. This strategy might seem reasonable if several players have indicated strong hands by betting heavily before the draw. When determining whether a 2-pair hand ranks relatively high or low, it is good to keep in mind that the *median* 2-pair hand is $JJ552$; a hand like $TT998$ may look attractive—all those pips in sight—but it's in the bottom half of all 2-pair hands.

When drawing 3 cards to a pair, there are $\binom{47}{3} = 16,215$ different combinations possible. Suppose that the original hand is $xxyyz$, with $x > y$. Certain final hands are easy to count: 4 of a kind must be $xxxxa$, where a can be any card including a y or z, so the number of 4-of-a-kind hands is

$$\binom{2}{2} \cdot \binom{45}{1} = 45.$$

There are two options for a full house: $xxxaa$ or $aaaxx$. The first of these may be drawn in

$$\binom{2}{1} \cdot \left[\binom{10}{1}\binom{4}{3} + 1 + \binom{3}{2} \right]$$

ways, where the 3 terms in brackets enumerate the general case for a, the case $a = y$, and the case $a = z$ respectively. The second case may be drawn in

$$\binom{10}{1}\binom{4}{3} + 1$$

ways, where the $+1$ term covers the case $xxzzz$. We have a total of 169 full houses, so the probability of a full house is approximately $\frac{1}{96}$, considerably less that the probability of drawing a full house with a 1-card draw to 2 pairs.

There is some compensation when we count 3-of-a-kind hands, which are impossible when drawing 1 card to $xxyy$. This hand must contain 3 xs, and there are 4 cases to consider. All 4 include an initial factor of $\binom{2}{1}$, representing the number of ways to choose a third x from the 2 remaining in the deck.

- $\boldsymbol{xxxab}$. This is the most common case, and may be drawn in
$$\binom{2}{1} \cdot \binom{10}{2} 4^2 = 1440$$
ways.
- $\boldsymbol{xxxyb}$. There are
$$\binom{2}{1} \cdot \binom{2}{1} \cdot \binom{10}{1} \binom{4}{1} = 160$$
ways to draw this hand.
- $\boldsymbol{xxxzb}$. This case is similar to the previous one. It can appear in
$$\binom{2}{1} \cdot \binom{3}{1} \cdot \binom{10}{1} \binom{4}{1} = 240$$
different ways.
- $\boldsymbol{xxxyz}$. If both discarded ranks reappear, there are
$$\binom{2}{1} \cdot \binom{2}{1} \cdot \binom{3}{1} = 12$$
further 3-of-a-kinds.

Adding shows that 3 of a kind may be drawn in 1852 ways. The probability of 3 of a kind,
$$\frac{1852}{16,215} \approx \frac{1}{8.76},$$
exceeds the $\frac{1}{11.75}$ probability of improving 2 pairs to a full house by a 1-card draw.

The chances of drawing into 2 pair (again) or of ending up with only 1 pair involve the consideration of multiple subcases as were seen in computing the chance of 3 of a kind, and are left to the reader to compute. Table 3.1 shows all of the possible outcomes with their approximate probabilities.

TABLE 3.1: Possible outcomes when drawing 3 cards to a pair after discarding a pair.

Final hand	**Ways to draw**	**Probability**
4 of a kind	45	$^3/_{1081}$
Full house	169	$^1/_{96}$
3 of a kind	1852	$^1/_{8.76}$
2 pairs	2629	$^1/_{6.17}$
1 pair	11,520	$^1/_{1.41}$

With a low probability of improving 2 pairs to a full house whether drawing 1 card or 3, a player may occasionally try bluffing by standing pat and drawing no cards. This is intended to give the impression that the hand is stronger than it is—perhaps a straight, flush, or full house—and drive out opponents after the draw.

Example 3.4. Consider the following hand:

$$K\spadesuit\ 8\spadesuit\ 7\heartsuit\ 6\spadesuit\ 5\spadesuit,$$

which contains both a 4-card straight and a 4-card flush. Is it better to draw 1 card to the flush or to the open-ended straight?

There are two competing answers to this question. At least one of them must be "neat, plausible, and wrong", to borrow again from Mencken.

1. Since a flush beats a straight, draw to the flush.
2. Since a flush beats a straight, there are more straights than flushes. Draw to the more numerous hand, since this hand is very weak if it doesn't improve in the draw.

We can resolve this by computing the success probabilities. There are 9 spades left among the 47 cards in the deck, while 8 cards—all of the 9s and 4s—will fill out the flush. The chance of drawing into the flush is greater by $\frac{1}{47}$, and so discarding the 7♡ and drawing is the better choice. ■

With this in mind, one might wonder where the flaw in the second line of thinking above is. While it is true that flushes outrank straights, it is more difficult to draw a 4-flush than a 4-straight in 5 cards, as we saw on page 58 when considering Canadian stud poker [25]. Arguably the harder part of drawing a flush is complete, and that, combined with the hand rankings, makes pursuit of the flush the better path.

The decision to draw 1 card to a 4-flush should take the pot odds into consideration as well. If the size of the pot is not at least 4¼ times the cost to call, even pursuing the flush is not viable. If there is no 4-flush to consider, drawing 1 to a 4-card open-ended straight can be justified on expected-value grounds if the pot is at least 5 times the amount required to stay in.

Kickers

Another way to give an image of strength when drawing to a pair is to hold a third card—called a *kicker*—and draw 2 cards instead of 3. Often the kicker is an ace or other high card, in the hopes of drawing into a high 2-pair hand such as aces up. How does this affect the outcome of the hand?

Consider a 3-card draw to 9♢ 9♣, with $\binom{47}{3} = 16,215$ possibilities, compared to a 2-card draw to 9♢ 9♣ A♢, which can be completed in $\binom{47}{2} = 1081$ ways. In either case, there is 1 rank (9s) with 2 cards remaining, 3 ranks with 3 cards remaining, and 8 ranks with all 4 cards still in the deck. The ways to improve these hands are compared in Table 3.2.

TABLE 3.2: Probabilities of the possible outcomes when drawing to a pair vs. a pair plus a kicker.

Final hand	**Draw 3**	**Draw 2**
4 of a kind	.0028	.0009
Full house	.0102	.0083
3 of a kind	.1143	.0777
2 pairs	.1599	.1776
1 pair (no change)	.7129	.7354

Holding a kicker brings a slight increase in the probability of drawing into 2 pairs, but 2 pairs will not help if an opponent holds 3 of a kind. Otherwise, the chance of improving is better without the third card—except for the possible effect on other players who overvalue your hand. Another, minor, consideration is that if the deck has not been thoroughly shuffled between hands, a pair that was thrown in at the end of the previous hand might stay together through the shuffle. This might raise the chance of getting another pair when drawing 2 or 3 cards by a small amount.

Consider next the case where you hold 3 of a kind. The standard choice is to discard 2 cards in pursuit of a full house or 4 of a kind. It may be strategically desirable, however, to hold a kicker and so disguise your hand's strength, for your opponents may then think that you're drawing 1 card to a relatively weak 4-card flush or straight. If you draw 1 card to $xxxy$, the probabilities of improving are

$$P(4 \text{ of a kind}) = \frac{1}{47}$$
$$P(\text{Full house}) = \frac{3}{47}$$

for a total chance of improving the hand of $\frac{4}{47}$.

If you draw 2 cards to the triple, the possible improvements are

$$P(\text{4 of a kind}) = \frac{46}{\binom{47}{2}} = \frac{2}{47}$$

$$P(\text{Full house}) = \frac{10 \cdot \binom{4}{2} + 2 \cdot \binom{3}{2}}{\binom{47}{2}} = \frac{66}{1081}.$$

The chance of drawing into 4 of a kind is twice as great without the kicker, as one might expect; the chance of a full house is slightly higher when holding the kicker. The value of disguising one's hand and encouraging opponents with weaker cards to stay in the hand is trickier to quantify.

Wild Cards

Since all of the cards in a hand of draw poker remain concealed until the showdown, adding one or more wild cards to the deck allows for higher-ranked hands while retaining an element of strategy that is diminished when a wild card is revealed in stud poker.

Consider a 7-handed draw poker game with a single wild joker. The probability that one of the 7 players holds the joker after the initial hands are dealt can be computed by looking at the 7 hands as a set of 35 cards. The probability is

$$\frac{1 \cdot \binom{52}{34}}{\binom{53}{35}} = \frac{35}{53} \approx .6604,$$

so nearly 2 hands out of 3 will see a player holding the joker early on.

If there are 2 jokers, the probability that someone has a joker off the top is found using the Complement Rule:

$$\begin{aligned} P(\text{At least 1 joker}) &= 1 - P(\text{No jokers}) \\ &= 1 - \frac{\binom{52}{35}}{\binom{54}{35}} \\ &= \frac{140}{159} \approx .8805. \end{aligned}$$

The chance of a joker on the deal rises to approximately 8 times in every 9 hands.

The power of a wild card in draw poker is enhanced when making a 1-card draw to a 4-card straight that includes a joker. When drawing 1 card to the 4-card straight 8♣ 7♠ 6♠ 5♡, the chance of completing the straight is $\frac{9}{48}$: there are four 9s, four 4s, and the joker remaining to fill out the straight.

If a joker replaces the 8♣, the number of cards that can complete a straight rises to 16. Any 3, 4, 8, or 9 turns this hand into a straight, so the probability of a straight is $\frac{1}{3}$.

The Bug

Consider a draw poker game using a single joker that treats it only as a bug. The number of hands increases, from $\binom{52}{5} = 2,598,960$ to $\binom{53}{5} = 2,869,685$. It's possible to hold 4 cards to a straight flush, including the bug, where the chance of improvement beyond a low pair exceeds 50%. Using W to represent the wild joker, if you discard the 9♢ and draw 1 card to

$$W\ 6♣\ 5♣\ 4♣,$$

then there are 22 cards which will give you a straight, flush, or straight flush. Additionally, drawing the A♠, A♡, or A♢ will give you a pair of aces, using the bug. There are thus 25 of the 48 unknown cards that will give you at least a pair of aces, so the chance of significant improvement is approximately .5208.

Draw poker with a bug calls for revisiting the question of holding a kicker when drawing. If the kicker is not an ace, the potential for improvement drops considerably. However, the chance of completing a 4-card straight rises if the bug is in-hand. The 4-card straight 5♢ 4♠ 3♠ Bug can be filled out to a straight by any 7, 6, 2, or ace, making the probability of a straight with a 1-card draw $\frac{16}{47} \approx .3404$. This is double the chance of completing a 4-card outside straight with a 1-card draw. The added advantage is much smaller when drawing 1 card to a 4-card flush including the bug, since there's no useful flexibility in how the bug may be designated—you can call the bug any rank you might wish, but the suit is not so malleable; there is only one choice for the suit when drawing to an incomplete flush.

3.3 Lowball Draw Games

Ace to 5 Lowball

Deck composition:	53 cards
Hand size:	5 cards
Betting rounds:	2

Ace to 5 or California lowball is customarily played with a fully wild joker and does not count straights or flushes as high hands, so the lowest possible hand is $5432A$: a *wheel*. Aces always rank low in this game.

Lowball hands are frequently referred to by the highest card they hold; thus $7643A$ would be called a "Seven". For convenience, we shall use "Seven" to refer to a hand ranked 7-high or less and reserve the use of "7" for a single card, and similarly for other high-card hands. A common rule in ace to 5 lowball is that a player holding a Seven or lower cannot check and must bet it. This rules out the possibility that a player with an excellent low hand can sandbag and draw players in before raising.

How many hands does this rule affect? We shall count the number of 7 or better hands by dividing them into two subsets: with and without the joker. We assume that a joker must be valued as the lowest possible card not present in the player's hand, so a hand like $W652A$ would call the joker a 3 and thus qualify as 7 or lower. Calling the joker an 8 in an attempt to evade the "must bet a Seven" rule would not be permitted; if discovered, the player could be subject to penalties after the showdown [11].

A jokered Seven must have the form $Wxyza$, where each letter represents a different card rank of ace through 7. There are

$$\binom{7}{4} = 35$$

ways to choose the ranks, from 7654 down to $432A$. Multiplying by the $4^4 = 64$ choices for suits gives 8960 jokered Sevens.

For Sevens without a joker, we simply choose 5 ranks from ace through 7, multiplying again by 4^4 to account for suits. This results in

$$\binom{7}{5} \cdot 4^4 = 5376$$

Sevens without the joker. Before the draw, then, there are 14,336 Sevens, which represent .4996% of all hands.

We note that jokered Sevens exceed "natural" Sevens: their ratio is 5 to 3. This points up one advantage of holding the joker: lower hands are, in general, possible with it than without it. Consider wheels: the best possible lowball hand. There are 4^5 wheels without the joker, but $5 \cdot 4^4$ jokered wheels: 25% more. As we see in other poker games, adding wild cards means that better hands—whether "better" is defined as "higher" or "lower"—result on average.

A second advantage of holding the joker is that your opponents *don't* have it, which makes it harder for them to hold the lowest hand. This is important when considering the effects of the draw on a lowball hand. Drawing is always risky in lowball, since pairing any of your cards will, in all likelihood, take you out of contention for the lowest hand. Holding the joker gives some flexibility when drawing, since the value of the joker need not be determined until after the draw. While nothing can help you if you discard the $Q\spadesuit$ and draw a second 5 to the hand $W754$ with the joker read as an ace, drawing an ace turns this hand from $754A$ into $7542A$, with no pair.

Example 3.5. Suppose that you are dealt $W432K$ and wisely discard the king. What is the probability that you will make an Eight (8-high hand) or better?

Your final hand is an Eight or better if your new card is an ace or a 5 through 8. There are 20 such cards left among the 48 left in the deck, making the probability of an Eight or better $\frac{20}{48} \approx .4167$. ■

Deuce to 7 Lowball

Deck composition:	52 cards
Hand size:	5 cards
Betting rounds:	2

By contrast with California lowball, deuce to 7 or Kansas City lowball forgoes the joker and regards straights and flushes as high hands. Additionally, aces always count high. As a result, the lowest possible hand is 75432 with at least 2 suits represented; a wheel counts as a straight and a Six such as $6432A$ is read as an ace-high hand. The probability of a dealt 75432 that's not a flush is

$$\frac{4^5 - 4}{2,598,960} = \frac{1020}{2,598,960} = \frac{1}{2548}.$$

This is less than half the $\frac{1}{1246}$ chance of a dealt wheel in Ace to 5 lowball.

Example 3.6. Since flushes count as high hands in deuce to 7, one-card draws to 4 suited low cards such as $7642\diamondsuit$ are riskier than the same draw to an unsuited hand. The probability of a Seven when drawing to $7642\diamondsuit$ is $\frac{6}{47}$, since the $3\diamondsuit$ and $5\diamondsuit$ make the final hand a flush rather than a Seven. A one-card draw to an unsuited 7642 has probability $\frac{8}{47}$ of yielding a Seven. The situation is even worse if the four cards also admit the possibility of a straight, such as $7654\spadesuit$. ■

One strategy tip offered to deuce-to-7 players is never to draw 2 cards unless holding a deuce [11]. In addition to the inherent value of the lowest card in a lowball game, this is in part because, as we noted above concerning the joker in ace-to-5 lowball, holding a deuce means that your opponents cannot hold it. There are

$$\binom{12}{2} \cdot 4^2 \cdot 4 = 4224$$

combinations of 3 cards containing one deuce and no pairs, since one would not draw 2 cards to any hand containing a pair in any lowball game. It is, of course, possible to break up a pair to draw 3 cards, as when dealt $2\spadesuit\ 2\heartsuit\ 9\diamondsuit\ J\clubsuit\ 7\diamondsuit$: discarding a deuce and the $J\clubsuit$ gives a two-card draw with the possibility of drawing into a Nine. Moreover, discarding a deuce here means that there are two deuces that your opponents cannot hold.

Triple Draw Lowball

Deck composition:	52 cards
Hand size:	5 cards
Betting rounds:	4

In *triple draw lowball*, players compete to build the lowest possible 5-card hand, with 3 opportunities to discard and draw. A betting round follows the initial deal and each round of drawing. The game may be played under either ace-to-5 or deuce-to-7 rules for valuing hands and aces. In ace-to-5 lowball, the best possible hand is the wheel $5432A$; in deuce-to-7, it's 75432. The multiple draws effectively reduce the maximum number of players in a single hand to 6, assuming that players make sensible decisions about folding unpromising hands and drawing reasonably. (If the deck is dealt out and discards are recycled, it is highly unlikely that many low cards will be available, making drawing under such circumstances less successful in improving hands.)

Strong hands in triple draw lowball are typically 7-high or better; an 8-high hand is marginal. One strategy guide for deuce-to-7 triple draw lowball advises "Don't leave the gate without a deuce," meaning that players should fold an initial hand containing no deuces [12]. This is applicable whether playing ace-to-five or deuce-to-seven: whether aces count as low or high, deuces are always a good low card. Following this advice restricts the number of playable starting hands to

$$\binom{52}{5} - \binom{48}{5} = 886,656,$$

just more than one-third of all hands. While this includes hands containing more than one deuce, there are two advantages to holding multiple deuces at the start of a hand:

- Extra deuces can be disposed of by discarding.
- Every deuce that you hold is a deuce that your opponents cannot hold. This gives you additional knowledge that may prove useful as the hand progresses.

Consider a five-handed game. The probability that no player holds a deuce among their initial hands is

$$\frac{\binom{48}{25}}{\binom{52}{25}} = \frac{54}{833} \approx .0648.$$

This unlikely circumstance does mean that the chance of drawing a deuce is high, but there's no good way to recognize this in the rare event that it happens.

Another guide to drawing in deuce-to-7 triple draw lowball, once a player holds a deuce, is to be wary of drawing to a deuce and a 6, for fear of completing a straight. If you draw 2 cards to a hand of $236, 246$, or 256, it is impossible to draw into 75432 and entirely possible that the draw yields 65432. The probability of a straight is

$$\frac{4 \cdot 4}{\binom{47}{2}} \approx .0148,$$

which, while small, is deadly.

At the same time, drawing 1 card to an 8-high holding is only viable if the 4 cards hold no straight possibility.

Example 3.7. In a deuce-to-7 game, suppose that you hold $JT986$ against an opponent's 7432, with one draw remaining and no risk of flushes. With no consideration of any cards that have been discarded, what is your probability of winning if you stand pat?

You win if your opponent draws a queen, king, or ace, or pairs any of her cards. There are 24 of these cards among the 43 we do not know, so the probability of a win for you is $\frac{24}{43} \approx .5581$. This jack-high hand is a better than even-money favorite. ■

In ace-to-5 lowball, the best possible hands are somewhat lower than in deuce-to-7 since the ace ranks low and straights and flushes do not count as high hands. The value of an 8-high hand diminishes greatly.

3.4 Italian Poker

Deck composition:	32–40 cards
Hand size:	5 cards
Betting rounds:	2

Italian poker is a variation on 5-card draw with several rule changes that affect the ranking of hands. The size of the deck is determined by the number of players: 6 players use a 40-card deck (5 through ace), 5 players use 36 cards (6–ace), and 4 players use 32 cards (7–ace). In each of the 3 deck configurations, flushes are rarer than full houses.

An easy way to compute the deck size is to subtract the number of players from 11, which yields the rank of the lowest card in the stripped deck [100].

Ties are broken in Italian poker by ranking the suits. Hearts rank highest, followed by diamonds, clubs, and spades. There are several ways that this order is interpreted to break ties in gameplay [100].

- A tie among straights is broken by looking at the suit of the highest card. 7♣ 6♣ 5♢ 4♢ 3♠ beats 7♠ 6♡ 5♡ 4♡ 3♡, for example.
- With tied 2-pair hands—$AAQQ2$ vs. $AAQQ2$, for example—the winning hand is the one with the heart in the higher pair.
- Tied 1-pair hands look first to the odd cards, in order, with the higher rank winning. If two players are tied with the same ranks in the odd cards, such as two $TTJ87$ hands, the pair containing the heart wins.
- Tied high-card hands are resolved by looking at the suits of the high cards.

  Some players choose to look at the suits of the cards beyond the pair or the highest card in breaking ties between pairs or among high-card hands. If two players both hold $QJ743$, for example, the queens are deemed to establish the tie, and it's broken by looking at the suits of the jacks.

Using this order of suits makes a royal flush in hearts an unbeatable hand. In order that there be no unbeatable hand, Italian poker further divides straight flushes into 3 categories:

- A *minimum* straight flush is the lowest possible straight flush. With 4 players dealing 32 cards, the minimum straight flush is $T987A$ of one suit. Minimum straight flushes number 4, and always include the ace as the low card, regardless of the size of the deck.
- Royal flushes—$AKQJT$, regardless of the number of cards in the deck—are called *maximum* straight flushes. As with minimum straight flushes, there are 4 of these in each deck.
- All other straight flushes are called *medium* straight flushes. The number of medium straight flushes depends on the deck. A deck with r ranks, $r \geqslant 8$, has $4(r-5)$ medium straight flushes.

Straight flushes are then ranked in a nontransitive way. Medium straight flushes beat minimum straight flushes, and maximum straight flushes beat medium straight flushes, but a minimum straight flush beats a maximum straight flush. In this respect, straight flushes are like the outcomes of a game of rock/paper/scissors: A beats B and B beats C, but then C beats A.

Example 3.8. Using this rank order in a 6-player game, the hand $\clubsuit AKQJT$ ranks higher than $\heartsuit JT987$, which beats the minimum straight flush $\spadesuit 8765A$—but $\spadesuit 8765A$ beats $\clubsuit AKQJT$. ■

The net result of this classification system is that any straight flush can be beaten, and thus that no hand is unbeatable.

While the rules of Italian poker eliminate the possibility of 2 hands tying, it is possible for the showdown to result in a 3-way tie. If Robin holds the maximum straight flush $\diamondsuit AKQJT$, Chris' hand is the medium straight flush $\clubsuit T9876$, and Alex has $\spadesuit 8765A$, a minimum straight flush, then Robin beats Chris, Chris beats Alex, and Alex beats Robin. The hand is ruled a 3-way tie and all 3 players divide the pot. The suits are not considered because suits are only used as a tiebreaker, and these pairs of hands are not tied.

How likely is this obviously very rare event? In a 4-handed game dealt from a 32-card deck, the probabilities of these 3 hands being dealt to the players can be found using conditional probability. As with a standard deck, the only choice in a maximum (royal) flush is the suit, hence the probability is

$$P(\text{Maximum flush}) = \frac{4}{\binom{32}{5}}.$$

The presence of a maximum flush in someone else's hand means that there are only 3 suits in which another player may draw a minimum flush, $A789T$ here, since the ace and 10 are both needed for a minimum flush in a 32-card deck. There are 5 cards accounted for in the first player's maximum flush, so we have

$$P(\text{Minimum flush} \mid \text{Maximum flush}) = \frac{3}{\binom{27}{5}}.$$

Finally, there are 3 remaining choices for the ranks of a medium flush: $KQJT9, QJT98$, and $JT987$, and 2 suits left to choose from (two 10s are gone). Combining this information with the fact that only 22 cards remain to build a hand gives

$$P(\text{Medium flush} \mid \text{Maximum flush and minimum flush}) = \frac{6}{\binom{22}{5}}.$$

Multiplying shows that the probability of this triple tie is about 1 in 5.95 billion. Since Italian poker is a draw game, the actual probability will be slightly more than this, but not so much that a 3-way tie is likely to happen to a person in a lifetime of playing the game.

3.5 Draw Poker at Home

Many poker players got their introduction to the game in private games, playing with friends for very low or no stakes around a kitchen or dining room table. Some poker variants thrive in this informal environment, but have not made the leap to casinos or card rooms; we consider some of those games in this section.

Four-Card Double Draw

Deck composition:	52 cards
Hand size:	4 cards
Betting rounds:	3

Four-card double draw was proposed as a new poker game, “the first really new poker game in more than 150 years”, in 2006 [96]. As the name indicates, players have two rounds of discarding and drawing to collect the best possible 4-card hand. The formulas beginning on page 22 applied to 4-card hands give the hand frequencies in Table 3.3. The order of 4-card hands differs from

TABLE 3.3: Four-Card Double Draw: Hand count.

Hand	**Count**
4 of a kind	13
Straight flush	44
3 of a kind	2496
Straight	2772
2 pairs	2808
Flush	2816
Pair	82,368
High card	177,408

the order of 5-card hands: 3- and 4-of-a-kind hands rank higher and flushes rank considerably lower. One striking result is that four-card straights, 2-pair hands, and flushes are almost equally common.

Since the probability of a dealt high-card hand is 65.53%, greater than the corresponding probability of 50.12% for 5-card hands, discarding and drawing plays a much greater role in four-card double draw than in standard 5-card draw poker. Certainly one should never discard one pair from 2 dealt pairs, as was suggested above, because that hand is not going to become a full house—indeed, full houses don’t exist in 4-card double draw.

The probability of improving 2 pairs to 3 or 4 of a kind by discarding one pair is perhaps worth investigating. The chance of getting 3 or 4 of a kind when drawing 2 cards to a pair is

$$\frac{2 \cdot 46 + 1}{\binom{48}{2}} = \frac{93}{1128}.$$

The probability of missing this improvement on the first draw but hitting it on the second is

$$\frac{1035}{1128} \cdot \frac{2 \cdot 44 + 1}{\binom{44}{2}} \approx .0863;$$

adding gives a chance of .1688, just more than $\frac{1}{6}$, of reaching 3 or 4 of a kind. Since the lowest 2-pair hand, 3322, outranks nearly 97% of all other 4-card hands, standing on 2 pairs through both draw is probably the better choice.

There is no loss, of course, in trying to improve 3 of a kind to 4 of a kind with 2 draws. The chance of improving is

$$\frac{1}{48} + \frac{47}{48} \cdot \frac{1}{47} = \frac{1}{24}.$$

or $4\frac{1}{6}\%$.

Three Card Lowball with Triple Draw

Deck composition:	52 cards
Hand size:	3 cards
Betting rounds:	4

Three Card Lowball with Triple Draw was proposed by its advocates as a worthy option for dealers seeking to inject some variety into their home games [83]. As the name suggests, players are dealt only 3 cards instead of 5 or 7, and have 3 opportunities to discard and draw cards as they seek to collect the lowest-ranked hand. A betting round follows the initial deal and each round of drawing, making 4 rounds of betting in all.

In this game, aces always count low, and straights and flushes do not count against the players. The lowest hand is therefore $32A$. The highest hand, and thus the least desirable, is 3 of a kind. There are 52 of these: Select any card from the deck, which may be done in 52 ways. This card identifies a unique 3-of-a-kind hand consisting of the other 3 cards of that rank. At the other end of the ranking spectrum, there are

$$\binom{13}{3} \cdot 4^3 = 18,304$$

high-card hands. This number includes straights and flushes, which play here as high-card hands. With these rules for hand rank, Table 3.4 shows the relative frequencies of the various hands. Nearly 5 out of every 6 hands, 82.8%,

TABLE 3.4: Three-Card Lowball Poker with Triple Draw: Hand count.

Hand	**Count**
Three of a kind	52
Pair	3744
High card	18,304

are high-card hands, making the race to the bottom a run through no-pair hands.

The excitement of Three Card Lowball comes in the opportunity to exchange cards in pursuit of a lower hand. There is, of course, always the risk that a draw will yield a higher card than the one discarded, or worse, form a pair with one of the cards retained.

Example 3.9. Suppose you are dealt $K\clubsuit\ J\diamondsuit\ 6\spadesuit$. You are allowed to draw up to 3 cards, and opt to discard the king and jack. You draw 2 cards, winding up with $9\diamondsuit\ 6\spadesuit\ A\spadesuit$, which is a considerable improvement. On the second draw, you discard the $9\diamondsuit$ and receive the $3\heartsuit$.

When contemplating your choice for the third draw, you note that if you discard the 6, any 7 or higher, as well as any 3 or ace, will make your hand worse. Of the 46 unknown cards that remain, 31—more than half—give you a higher hand than the $63A$ you hold. The best choice is to stand on your $63A$ and hope that it's low enough.

Your hope is justified: the only hands that can beat you are $62A$, any 5-high or 4-high hand, or $32A$. ∎

How are the high-card hands distributed?

This question brings to mind the function $F(x)$ for 5-card hands computed on page 25. Since straights and flushes don't count as high-card hands, all we need to do to count the hands with a high card ranking x is count the number of sequences with x as the highest card and multiply by $4^3 = 64$, to account for the choices of suits. With the jack, queen, and king corresponding to $x = 11, 12, 13$ respectively, there are $\binom{x-1}{2}$ ways to select 2 ranks lower than x, and so the number of high-card hands with high card x is

$$\binom{x-1}{2} \cdot 64 = 32(x-1)(x-2).$$

There are 4224 king-high hands, and this number decreases down to 64 3-high hands.

Three Card Lowball has been adapted, with fewer draws, into a casino table game where players compete against a single dealer. See page 260.

Caro Dots

Deck composition:	52 or 53 cards: 1 optional joker
Hand size:	5 cards
Community cards:	None
Betting rounds:	2

Caro Dots was a draw poker variation designed by gaming writer Mike Caro in the early 1980s as an instructional example to illustrate certain fine points of poker analysis which would also be legal in California card rooms. While more suited for home play, perhaps, there is no reason why this game could not be adapted to a modern casino's poker room.

Caro Dots uses a standard 52-card deck. A joker may be added at the players' discretion. The object of the game is to hold the highest number of points ("dots") in a 5-card hand, with one round of discarding and drawing permitted. Aces through 6s are the point-scoring cards; each one is worth its face value. The optional joker is worth 7 points if used. Sevens and higher cards are worth no points.

Example 3.10. The hand 8♠ 5♡ T♡ 2♡ A♡ contains 3 scoring cards, and is worth 8 points. ■

Without a joker, the Caro Dots equivalent of a royal flush is a 29-point hand consisting of four 6s and a 5. There are 4 of these in a deck, so the probability of a 29 is the same as that of a royal flush. With the joker added, there is a unique 31-point hand: the joker and four 6s. Since the joker has been added to the deck, the probability of a 31-point hand is

$$\frac{1}{\binom{53}{5}} = \frac{1}{2,869,685}.$$

With or without the joker, though, only one of these maximum-point hands may occur in a single deal, and so the hand is unbeatable.

A hand of Caro Dots is played like a hand of 5-card draw poker: Each player is dealt 5 cards, and a round of betting follows. Players still in the pot then have a chance to discard some of their cards and draw replacements. After a second round of betting, the player with the most points wins the pot. Ties are broken in favor of the player with more points on red cards. (Sometimes, a game needs an arbitrary rule.)

Example 3.11. This rule doesn't break all ties, though. If one player holds

8♠ 6♠ 6♢ 2♡ A♠

and a second player has

Q♣ 5♡ 4♣ 3♢ 3♣,

then both hands score 15 points with 8 points on red cards, and so the hand ends in a draw. ■

Example 3.12. In a joker-free game, what is the probability of receiving an initial hand containing 0 points?

There are 7 ranks, or 28 cards, with zero point value—slightly more than half the deck. Before computing the exact probability, we shall try a quick approximation and see how close we come. Since the number of zero-point cards is close to half the deck, we estimate that the chance of drawing one is .5. The probabilities change as cards are dealt, of course—successive cards are not independent—but if we assume independence as a convenience, the probability of a hand scoring 0 is approximately

$$\left(\frac{1}{2}\right)^5 = \frac{1}{32} = .03125.$$

Figuring more exactly, the probability of being dealt 5 zero-point cards is

$$\frac{\binom{28}{5}}{\binom{52}{5}} = \frac{98,280}{2,598,960} = \frac{9}{238} \approx .0378.$$

This is reasonably close to our estimate above, and is at any rate less than 4%.

If the joker is included, the approximation above is unchanged, and the exact probability drops to .0342, so our quick approximation of .03125 is even closer to the exact value. ■

Strategy for Caro Dots is different from, and easier than, standard draw poker strategy, because a player is not trying to match any sort of pattern when drawing cards. The goal is simply to collect as many points as possible, so the only factor to be considered when discarding is a hand's potential for improvement by accumulating more points. If there is no joker, the deck contains 84 points, so an average card is worth $\frac{84}{52} \approx 1.6154$ points. Since this is more points than an ace, a simple and sensible rule for playing Caro Dots is to discard all aces as well as all zero-point cards [15]. In converting Caro Dots to a carnival game, where all players face off against a single dealer hand, this would serve well as the *house way*: the prescribed rule by which the house must play its cards.

This presumes, of course, that your hand is strong enough to justify not folding in the first round of betting. How many points are necessary to rate a hand as "strong" and bet it through to the draw? An average 5-card hand contains just over 8 points, suggesting this as a possible minimum value for betting on the first round.

However, the distribution of points within a hand matters. There's a qualitative difference between an 11-point hand like

$$7\spadesuit\ 3\diamondsuit\ 2\diamondsuit\ 4\diamondsuit\ 2\heartsuit$$

and one like

$$K\diamondsuit\ 9\clubsuit\ 9\diamondsuit\ 6\clubsuit\ 5\heartsuit.$$

The first hand only has one opportunity to swap a zero-point card for a higher card, while discarding 3 cards from the second hand is the obvious call without the need for complicated reasoning. The second hand thus has two more chances to improve. The average score of the first hand after drawing would be 12.6 points; the second hand's final score would average 14.8.

A guide for the Caro Dots player would take both the score of a hand and its potential for improvement into account.

Example 3.13. If we adopt a simple strategy of "Discard all cards worth 1 point or less", what is the probability of an initial hand that draws no cards?

Such a hand would contain 5 cards that are all 2s through 6s, and would have a minimum value of 11. The probability of such a hand, without a joker in play, is

$$\frac{\binom{20}{5}}{\binom{52}{5}} \approx \frac{1}{168}.$$

If the joker is used, the probability is about 1 in 141. ■

An advanced version of Caro Dots is *Caro Dots Accelerated*, a more volatile version of the game with the potential for higher scores and destructive draws. This game gives limited power, for both good and bad effects, to the face cards. A singleton face card is called an *accelerator*, and holding one doubles the value of all scoring cards of the same suit in the hand. If the joker is used, it cannot be doubled by any face card [16].

Example 3.14. The hand $2\spadesuit\ 2\clubsuit\ J\clubsuit\ A\diamondsuit\ 6\spadesuit$ would be worth 13 points, as the $2\clubsuit$ counts double its face value due to the jack acting as accelerator. ■

The highest hand in Caro Dots Accelerated scores 36 points without a joker: $\clubsuit Q6543$, for example. If a joker is used, a hand like Joker$\clubsuit Q654$ scores 37. There are several ways to deal a 36-point hand, so it's possible for two or more players to tie at 36. Since the maximum hand of 37 must use the only joker, there can be no ties at 37 points [16].

Face cards are deadly in combination, though. A hand with more than one face card scores 0 points, so drawing to a hand with one face card incurs some risk of drawing a second face card and turning a strong hand into a loser. On a three-card draw to the 10-point hand $J\spadesuit\ 5\spadesuit$, with 3 non-face cards discarded, the probability of drawing a second face card and zeroing out the hand is

$$1 - P(\text{No face cards}) = 1 - \frac{\binom{36}{3}}{\binom{47}{3}} \approx .5597.$$

High probabilities like these are part of the challenge that a Caro Dots Accelerated player faces.

3.6 Poker Dice

Poker dice are special 6-sided dice bearing the images of playing cards from 9 through ace. Suits, if present, are not typically considered in games played with these dice. Five poker dice are shown in Figure 3.1.

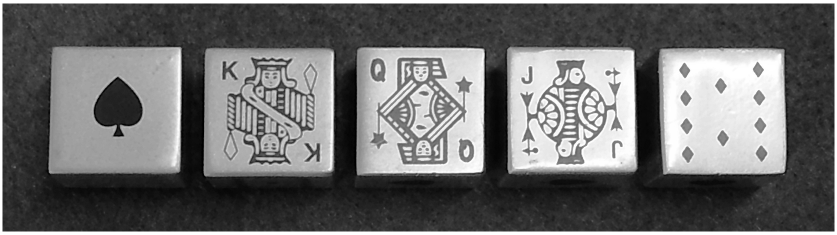

FIGURE 3.1: Poker dice.

Games played with poker dice can also be played with ordinary 6-sided dice. When this is done, ⚀ (ace) is usually the highest-ranked side, followed by ⚅, ⚄, and so on down to ⚁.

Poker dice games are frequently played as a variation on draw poker: Players roll all 5 dice to create a poker hand, and then may re-roll any number of the dice up to twice more in an effort to improve their hand. This is similar to the rules for the parlor game Yahtzee, which is also played with 5 dice. The dice are often shaken in a cup and rolled; this is thought to prevent dice mechanics from controlling their shots to bring up or avoid certain numbers.

Betting at poker dice is simpler than in other forms of poker: each player typically antes a fixed amount to a pot and there is no further wagering. The player throwing the highest hand wins the pot [120]. There are $6^5 = 7776$ possible poker dice hands, ranging from 5 of a kind down to high card. Since the individual dice are independent, counting the frequency of various hands is simple. As there are no suits to speak of, there are no flushes of any sort.

- There are 6 ways to throw 5 of a kind: one for each face. As five of a kind is the best hand, 5 aces is unbeatable, although it can be tied.
- Since there are no suits, the next highest hand is 4 of a kind. There are 6 choices for the quartet, $\binom{5}{4} = 5$ ways to choose the 4 dice that match, and 5 ways to roll the nonmatching die, for a total of 150 four of a kinds.
- Straights are of 2 types: $AKQJT$ and $KQJT9$. The ranks are predetermined, so all that we need to do is account for how 5 ranks may be arranged among the 5 dice. Order matters here—thinking of the dice as different colors may make this clear—so each set of ranks can appear in ${}_5P_5 = 120$ ways, giving 240 straights. In some settings, straights are not recognized, and play as high-card hands.
- For full houses, the triple may be chosen in 6 ways and distributed among 3 dice in $\binom{5}{3} = 10$ ways. This locks in the 2 dice that form the pair, and there are 5 ways to set their rank. Multiplying gives 300 ways to roll a full house. Full houses lose to straights in a game of poker dice.
- Counting 3 of a kinds starts by counting the 60 ways to roll the triple, which is calculated in the full house computation above. The nonmatching cards may be chosen in $\binom{5}{2} \cdot 2 = 20$ ways: the factor of 2 allows us to account for the different ways to assign 2 ranks to 2 different dice. We have 1200 three-of-a-kind hands.
- Following the reasoning already demonstrated allows us to see that there are
$$\binom{6}{2} \cdot \binom{5}{2} \cdot \binom{3}{2} \cdot 4 = 1800$$
2-pair hands.
- Hands containing a pair occur in
$$\binom{6}{1} \cdot \binom{5}{2}\binom{5}{3} \cdot {}_3P_3 = 3600$$
ways. Here, the factor of ${}_3P_3$ accounts for the number of ways to assign the 3 nonmatching faces to 3 dice.
- We rank high-card hands at the bottom, even though there are fewer of those than of some other hands. A high-card hand must be $AQJT9, AKJT9, AKQT9$, or $AKQJ9$. For each choice of ranks, there are ${}_5P_5 = 120$ ways to arrange the dice, so there are 480 high-card hands—720 if straights are counted only as high-card hands.

Example 3.15. The first roll of a set of poker dice is $TTAQ9$. Is it better to hold the ace as a kicker or hold on to the two 10s and reroll 3 dice?

If the player rerolls 3 dice, holding only the 2 tens, there are 216 possible results. They are sorted in Table 3.5.

TABLE 3.5: Outcomes when rolling 3 poker dice to TT.

Hand	Probability
5 of a kind	$1/216$
4 of a kind	$15/216$
Full house	$20/216$
3 of a kind	$60/216$
2 pairs	$60/216$
1 pair	$60/216$

It is interesting to note that the probabilities of 3 of a kind, 2 pairs, and 1 pair are all the same.

When holding the ace and rerolling 2 dice, there are 36 possible ways for the dice to fall. The final hands and their probabilities are listed in Table 3.6.

TABLE 3.6: Outcomes when rolling 2 poker dice to TTA.

Hand	Probability
4 of a kind	$1/36 = 6/216$
Full house	$3/36 = 18/216$
3 of a kind	$8/36 = 48/216$
2 pairs	$12/36 = 72/216$
1 pair	$12/36 = 72/216$

In each case, the median final hand is 2 pairs. Comparing the two tables reveals the following:

- Discarding the ace makes 5 of a kind possible, though unlikely.
- The chance of rolling into 4 of a kind increases when rerolling 3 dice—this is because 4 of a kind starting with TT must be 4 tens, and 3 dice gives a better chance of rolling 2 more 10s than 2 dice do.
- It is slightly more likely to roll a full house or 3 of a kind when starting with TT rather than TTA.
- As we saw in the analogous case in draw poker, holding the kicker slightly increases the probability of rolling 2 pairs.
- The chance of no improvement—the second roll resulting only in another pair of 10s—is higher when holding the ace.

The conclusion is clear: As when playing draw poker, holding a kicker results in a lower final hand in general. ■

Since players play out their hands to the conclusion in order, each player knows what hands her predecessors have rolled, and thus knows what she must beat. This gives a great advantage to players going later in the game's rotation.

Another game often played alongside poker dice in the early 20th century is a solo game played against a willing bartender or cigar store proprietor [23]. Gamblers would put up a stake and select a number from 1 to 6. They would then roll 5 dice 25 times, seeking to roll 25 or more of their chosen number. This could also be played with standard poker dice, with the gambler choosing a card rank.

This is a binomial experiment once the target number has been chosen; the 125 independent trials (die rolls) can be separated into "Is the target" and "Is not the target". Let X count the number of times the target number is thrown. The probability of winning this bet is

$$P(X \geqslant 25) = \sum_{k=25}^{125} \binom{125}{k} \cdot \left(\frac{1}{6}\right)^k \cdot \left(\frac{5}{6}\right)^{125-k} \approx .1877.$$

If the payoff to a winner is 4–1, the game operator holds an advantage of 6.17%. A cigar store offering only 3–1 will keep almost 25% of the total wagered in the long run.

Other sets of poker dice use five octahedral (8-sided) dice and restrict the deck to the top 40 cards: 5s through aces in all 4 suits. Suits are important when using these dice. Calculation of possible hands requires careful attention to which cards appear on which dice, for while there are $8^5 = 32,768$ hands that can be rolled, many important combinations are impossible because they require 2 cards that appear on the same die. Five octahedral poker dice are shown in Figure 3.2.

FIGURE 3.2: 8-sided poker dice.

We can label these dice as A to E from left to right. The cards on each die are listed in Table 3.7.

Consideration of Table 3.7 reveals that a royal flush in spades or hearts is impossible, as the $K\heartsuit$ and $T\heartsuit$ are both on die C and the $K\spadesuit$ and $T\spadesuit$ both appear on die D. The probability of rolling a royal flush with these dice is then

$$\frac{2}{8^5} = \frac{1}{16,384}.$$

TABLE 3.7: Cards shown on the 8-sided poker dice from Figure 3.2.

Die	**Cards**							
A	$A\spadesuit$	$K\clubsuit$	$T\diamondsuit$	$9\heartsuit$	$8\clubsuit$	$7\spadesuit$	$6\heartsuit$	$5\diamondsuit$
B	$A\heartsuit$	$K\diamondsuit$	$Q\spadesuit$	$J\clubsuit$	$8\diamondsuit$	$7\heartsuit$	$6\clubsuit$	$5\spadesuit$
C	$A\diamondsuit$	$K\heartsuit$	$Q\clubsuit$	$J\spadesuit$	$T\heartsuit$	$9\diamondsuit$	$8\spadesuit$	$7\clubsuit$
D	$A\clubsuit$	$K\spadesuit$	$Q\heartsuit$	$J\diamondsuit$	$T\spadesuit$	$9\clubsuit$	$6\diamondsuit$	$5\heartsuit$
E	$Q\diamondsuit$	$J\heartsuit$	$T\clubsuit$	$9\spadesuit$	$8\heartsuit$	$7\diamondsuit$	$6\spadesuit$	$5\clubsuit$

This could be changed by swapping some cards, for example, switching the $K\spadesuit$ with the $8\heartsuit$ on die E, but that would break up several symmetries that are built into this set of dice:

- Each suit appears exactly twice on each die. Practically speaking, this means that the probability of completing a 4-die flush by rerolling the odd die will always be $\frac{1}{4}$, independent of which suit is involved and which die is rolled.
- Each rank is paired with an adjacent rank, and that pair appears together on 4 of the 5 dice.
- Within these pairs, cards are matched by color: diamonds always appear with hearts and spades with clubs. For example, the $6\diamondsuit$ and $5\heartsuit$ are both on die D, while the $6\heartsuit$ and $5\diamondsuit$ are together on die A.

This limited availability of royal flushes extends to straight flushes, which are also impossible in spades or hearts. Four of a kind is possible with any rank, due to the somewhat obvious and very reasonable property that no two cards of the same rank appear on the same die. Full houses are not always possible, since the paired ranks—$A/K, Q/J, T/9, 8/7$, and $6/5$—are all missing as a pair from one of the dice, and so cannot form full houses with each other.

The basic poker dice game is the same with these dice as it was with 6-sided poker dice: players roll all 5 dice and then have 2 optional opportunities to reroll some or all of the dice as they seek to improve their poker hand. Players must keep the assignment of cards to dice in mind as they make their strategy decisions if they want to get the best results.

Example 3.16. When holding $K\clubsuit$ $K\diamondsuit$ $K\heartsuit$ $A\clubsuit$ $6\spadesuit$, it is pointless to hold the ace as a kicker, rerolling only die E (bearing the $6\spadesuit$) in hopes of improving to a full house. This hand cannot improve past 3 kings with this strategy, since die E has neither an ace nor a king.

A better choice would be to hold the kings, rerolling both dice D and E. The chance of rolling into 4 of a kind is then $\frac{1}{8}$, while the chance of a full house is

$$\frac{3}{4} \cdot \frac{1}{8} = \frac{3}{32} \approx .0938.$$

■

3.7 Exercises

Answers to starred exercises begin on page 336.

Draw Poker Without Wild Cards

In Exercises 3.1–3.7, assume that the game is dealt from a 52-card deck, without a bug or joker.

3.1. Show that the following 4-card hands have equal probability of improving with a 1-card draw:

- 3 of a kind plus a kicker.
- An inside straight.

3.2.* Find the probability of improving a 4-card open-ended straight flush (such as ♡3456) by drawing 1 card.

3.3.* Consider the event of drawing 1 card to a four-flush in the hopes of improving to a flush.

a. Find the probability of missing the flush.

b. Find the probability of missing the flush 50 times in a row.

c. How many hands would you have to draw 1 card to for the probability of missing all of them to be lower than 50% for the first time?

3.4. Some home draw poker games allow a player to draw 4 cards if holding an ace, which frequently must be shown to all other players. Show that the probability that a 4-card draw to an ace, discarding 4 cards of different ranks, results in a pair of aces is approximately 1 in 5.

3.5.* Suppose that you draw 1 card to a two-pair hand.

a. Find the probability that you complete a full house.

Use the binomial distribution and the answer to part a. to compute the following probabilities.

b. P(Exactly 1 full house in 10 such draws).

c. P(10 full houses in 10 such draws).

d. P(0 full houses in 20 such draws).

3.6. Consider an initial hand including 3 queens, an ace, and an 8. Find the probability of improvement if you discard

a. The ace and the 8.

b. Only the 8.

3.7. In Example 3.9, find the probability that your 63A hand wins against a single opponent in the showdown.

Draw Poker With Wild Cards

3.8. In a game dealt with a single bug, find the probability that a 5-card flush is ace-high.

3.9.⋆ In a draw poker game dealt from a 53-card deck with a fully wild joker, find the probability that you are never dealt the joker in your initial hand during 3 hours of play. Assume that 40 hands are dealt per hour.

3.10.⋆ A *16-way straight* is a hand like Joker 8♢ 7♣ 6♠ 2♠. Discarding the deuce gives 16 cards that will fill this hand out to a straight. Find the probability of missing the straight every time when drawing to 10 consecutive 16-way straights. Assume a 53-card deck with a single joker

3.11.⋆ *Woolworth Draw* is a draw poker variant that pays tribute to the legendary American 5- and 10-cent store. In this game, all 5s and 10s are wild [128]. Opening the betting in this game requires that the player hold both a 5 and a 10 in his or her hand. It's not sufficient to hold 2 wild cards; both ranks must be represented. Find the probability that a player holds openers.

3.12.⋆ An extreme form of draw poker using 12 wild cards is *Dr Pepper,* named for the soft drink. A 1920s ad campaign for Dr Pepper used the line "Drink a bite to eat at 10, 2, and 4", listing times during the day when people's blood sugar might be low and in need of a pick-me-up. In line with this slogan, 10s, 2s, and 4s are wild in the game [5].

a. Construct the PDF for the number of wild cards, X, in a dealt hand.

b. Use the PDF found in part a. to find the probability of a hand containing no wild cards.

c. What is the average number of wild cards in a Dr Pepper hand?

3.13.⋆ Find the probability of a dealt initial hand of 5 aces in Dr Pepper.

3.14.⋆ In [24], an unusual game of Dr Pepper is described: 7-card stud with 10s, 2s, and 4s all wild, and a showdown sees players using more than 5 cards in crafting their final hands. Four players in the showdown successively reveal 5 aces, a 6-card straight flush, and 6 kings, before the pot is won by a player holding seven 5s. Find the probability of a 7-card Dr Pepper hand composed of seven 5s.

3.15.⋆ Find the average value of a Caro Dots hand dealt from a 53-card deck including a joker.

Proposition Bets

Some poker hustlers have offered informal *proposition* or *prop* bets, wagers unrelated to the play of the main game, to players. These bets are typically simple and based on the cards dealt to the player. All hands in this section are dealt from a 52-card deck.

3.16.* One prop bet is to offer 100–1 odds against the player holding a *pinochle*: the $Q\spadesuit$ and $J\diamondsuit$ [11]. A player opting in to this bet would put up $1 per hand, and would receive $100 from the hustler if dealt a pinochle.

a. Find the probability that a player's initial 5-card hand holds a pinochle.

b. Use this probability to compute the expected value of the bet. What edge does the hustler hold?

3.17.* Mike Caro described a prop bet he observed where a friend of his was offering a poker opponent 150–1 odds ($300 vs. $2) that the opponent would not get both red jacks in his initial hand before the draw. Caro pointed out that his friend had the worst of this prop bet [11].

a. Find the probability of holding both red jacks in 5 cards and compute the opponent's advantage.

b. Caro's friend had offered this bet without consideration of the possibility that his opponent held more than 2 jacks. Find the probability that a 5-card hand holding exactly 2 jacks contains the $J\heartsuit$ and $J\diamondsuit$ and use that probability to show that, under the (incorrect) assumption that this is the probability of being dealt both red jacks, the advantage lies with Caro's friend.

Poker Dice

3.18.* A variation on poker dice when ordinary 6-sided dice are used declares that deuces—that is, dice showing ⚁—are wild. With this rule in force, find the probability of throwing 5 of a kind on the first roll.

3.19.* In the poker dice game where the goal is to roll 25 of a kind, find the expected number of dice that show the target number.

3.20.* Consider the following adaptation of the "roll 25 of a kind" game to 8-sided poker dice. A player chooses a rank from 5 through ace and rolls the 4 dice bearing a card of that rank 25 times, for a total of 100 rolls.

a. Find the expected number of times that the chosen rank is rolled.

b. If the player must roll her selected rank 15 or more times to win, find the probability of winning.

c. Use the probability computed in part b. to find the house edge if winners are paid at 2–1.

Chapter 4

Texas Hold'em and its Variants

Texas hold'em (Figure 4.1) is arguably the most popular form of poker in the 21st century. Though hold'em dates back to well before 2001, its recent rise in popularity can be attributed to televised tournaments which use small cameras to reveal players' hole cards to a viewing audience. Additionally, televised hold'em games use computers to calculate and display the probability of each player's hand winning the pot by rapidly running through each possible finished game in real time and identifying the best hand—an exercise in experimental probability that enhances the experience for the viewers.

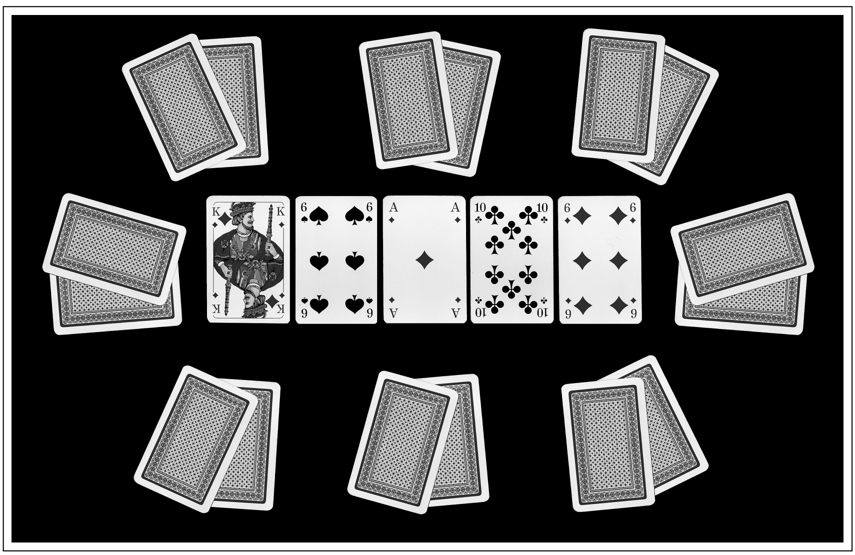

FIGURE 4.1: An 8-player game of Texas hold'em [110].

The game appears to have been created in Texas in the early 1900s, originally under the name Hold Me Darling, which was shortened to Hold Me and thence became Hold'em. In 2007, the Texas State Legislature officially recognized Robstown, Texas as the game's birthplace [101]. Oddly, Texas hold'em may not legally be played in public in the state of Texas. Some private card clubs operate in Texas, but their legal status is unclear. Patrons are frequently assessed a monthly or daily membership fee plus an hourly fee when actually gambling. These charges skirt the state's anti-gambling laws by replacing the

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rake and rendering the games private affairs that are not open to the public [124]. How long this quasi-legal status might last before Texas moves to shut down or regulate and tax private card clubs is not certain.

A 1968 article in *Life* magazine described a "new poker game" called Hold Me, which author A.D. Livingston predicted would replace stud poker as the dominant form of the game in America before the end of the 20th century [91]. If not exactly right, he was not far off. Hold'em arrived in downtown Las Vegas at the California Club and Golden Nugget before making its first appearance on the Strip at the Dunes in 1969. The game began its rise to prominence when it was designated the main event of the first World Series of Poker, held in downtown Las Vegas at Binion's Horseshoe in 1971.

Home poker games are often played "dealer's choice", where the deal rotates from player to player after each hand, and each dealer in turn may select the game that he or she deals. As a home game, the Texas hold'em dealer has a considerable advantage, since he bets last and can base his decision on the number of other players who have stayed in the game. In a casino's poker room, where the dealer merely distributes the cards and does not gamble, the dealer position is rotated among the players by placing a *dealer button* (Figure 4.2): a flat disk about the size of a casino chip, in front of the designated dealer.

FIGURE 4.2: Dealer button, as used in Texas hold'em.

The use of a dealer button allows the advantages and disadvantages of being early or late to act to be shared among the players. The button rotates around the table so that each player occasionally shares the advantage of being last to act. This button also identifies the players who must post the small and big blind bets (page 49) before the cards are dealt: they sit in the two seats immediately to the left of the button. The player to the left of the button posts the small blind and the player to his or her left posts the big blind. The blinds are the last to act in the initial betting round. If no one has made a bet after the hole cards have been dealt, the small blind may stay in the hand by "completing" his small blind to the amount of the big blind. The player making the big blind bet can stay in the hand without betting further if no one has raised in the first round.

Example 4.1. In the Main Event no-limit hold'em game at the World Series of Poker, the size of the blinds rises as the tournament progresses and players are eliminated. In 2023, every player started this tournament with 60,000 chips. With 10,043 players entering, over 600 million chips were in play. At the first level, comprising the first 2 hours of play, the blinds were 100 and 200 chips. Every 2 hours, the blinds increased, from 200 and 300 chips at level 2 up to 5 million and 10 million chips at the 47th and final level—if the tournament was not yet decided before that point. The increasing blinds were intended to move the tournament along more quickly by raising the size of pots and forcing higher action. Five levels were played per day until the field was narrowed to the final 9 players.

7% of the entry fees was kept by tournament organizers as a rake of sorts, leaving $93,399,900 in the prize pool [122]. 1508 of the entrants won a cash prize, ranging from a return of their $10,000 entry fee up to the top prize of $12,100,000. ■

For home games operating without a button, a fix is quite simple. On each round, the first player offered a chance to open the betting rotates counterclockwise around the table. Before the flop, the first player to bet is the one seated to the dealer's left, as is the case in standard poker room hold'em. After the flop, the second player on the dealer's left who remains in the hand bets first, and so on through betting rounds 3 and 4 [5]. As the first bet moves around the table, the advantage of being last to act moves right behind it.

At the Ocean's Eleven Casino in Oceanside, California, a game called *Position Poker* is Texas hold'em with the added rule that a second button, labeled "Position", is placed in front of the winner of the previous hand. Possession of the position button gives that player an extra advantage: the right of the last bet in each round of the current hand. Split pots are resolved by awarding the position button to the player with the highest-ranked suit. Spades are ranked highest, followed by hearts, diamonds, and clubs.

4.1 Basic Gameplay

Deck composition:	52 cards
Hand size:	7 cards
Community cards:	5
Betting rounds:	4

A legend holds that Texas hold'em was invented by a group of 20 cowboys who wanted to play poker but had only 1 deck of cards [75]. Those cowboys could play a hand of hold'em using only 45 of the 52 cards in a deck.

A hand of Texas hold'em begins when each player is dealt two hole cards face down. A round of betting follows. The dealer then burns a card (removes it from play without exposing it) and then deals 3 cards face up to the table. These cards are collectively called the *flop*, and are the first three *community cards* which may be used by any player to form their final hand.

Following a second betting round, another card is burned and a fourth community card, the *turn*, is dealt. Another round of betting is followed by another burned card and the fifth, final community card, the *river*, is dealt. The 5 community cards are often called the *board*. One final betting round takes place, and then the players who have not yet folded make the best possible hand from their hole cards and the 5 community cards, with the top hand winning the showdown and the pot.

Example 4.2. Suppose that two players bet through to the end of the hand. Chris's hole cards are $Q\clubsuit$ $8\clubsuit$ and Robin holds $T\spadesuit$ $9\heartsuit$. The community cards are as follows:

- The flop is $4\spadesuit$ $8\spadesuit$ $5\spadesuit$.
- The turn is the $9\diamondsuit$.
- The river is the $6\spadesuit$.

Chris' best hand is a pair of 8s, which loses to Robin's spade flush. ■

Only the top 5 cards in each player's hand matter in the showdown. If Terry's hole cards are $K\heartsuit$ $2\diamondsuit$, Sandy holds $K\clubsuit$ $5\heartsuit$, and the board is

$$A\diamondsuit\ K\spadesuit\ 4\spadesuit\ 4\diamondsuit\ 2\spadesuit,$$

the hand is a tie. Both players hold 2 pairs, kings and 4s, and both use the $A\diamondsuit$ on the board as their fifth card. The fact that Terry holds 3 pairs if the hands are extended to 6 cards is not used as a tiebreaker.

The probability of a royal flush in a 5-card game is 1 in 649,740. With 7 cards to choose 5 from, but far more 7-card combinations than 5-card ones, how does this probability change in Texas hold'em?

There are 4 royal flushes; each may be combined with any 2 of the other 47 cards in the deck. The probability of a royal flush is

$$\frac{4 \cdot \binom{47}{2}}{\binom{52}{7}} = \frac{1}{30,940}.$$

Royal flushes may be 21 times more likely, but there is a small chance,

$$p = \frac{4}{2,598,960},$$

that the 5 cards of the royal flush all appear on the board, generating a tie among all players who have not folded.

Hole Cards Considered

The best possible hole cards are a pair of aces. The probability of aces in the hole is

$$\frac{\binom{4}{2}}{\binom{52}{2}} = \frac{1}{221}.$$

This is also the probability of any specified pair, regardless of rank, in the hole.

Consider our group of 20 cowboys playing a hand of the game that they may have invented. What is the probability that none of them holds an ace in their hole cards?

We shall solve this problem for the general case of n gamblers and then set $n = 20$. A total of $2n$ cards will be dealt before the flop. The chance that no aces appear among $2n$ cards is then

$$P(n) = \frac{\binom{48}{2n}}{\binom{52}{2n}} = \frac{(n-26)(n-25)(2n-51)(2n-49)}{1,624,350}.$$

If $n = 20$, we have $P(20) \approx .0018$, or about 1 chance in 547. For the more usual 9-handed game, such as is played at the final table in the World Series of Poker, the probability that no one holds an ace in the hole is $P(9) \approx .1713$, about once in every 5.8 hands.

At the other end of the spectrum, the worst hole cards are a 2 and 7 of different suits: two low-ranked cards that are too far apart to readily fill in a straight and with no head start on drawing a flush. The probability of an offsuit 2–7 is

$$\frac{4 \cdot 4 - 4}{\binom{52}{2}} = \frac{12}{1326} = \frac{2}{221}$$

—twice as likely as a pair of aces.

One of the first books on Texas hold'em was published in 1976 by David Sklansky [140]. He listed 87 different hole-card holdings that he considered generally worth betting, ranking them from AA down to an unsuited 5 and 4. This list showed suited and unsuited cards separately, with suited hole cards ranking higher for their head start on a flush. For example, while an unsuited 5 and 4 ranks 87th on Sklansky's list, a suited 5 and 4 holds the 53rd slot.

Without accounting for the fact that a pair in the hole is rarer than unpaired hole cards, how many different combinations of hole cards—suited or unsuited—are possible? There are $\binom{13}{2} = 1326$ ways to draw 2 cards, but many of these combinations are indistinguishable for our immediate purposes. For example, if you hold a suited 5 and 3 (hand #65), the suit isn't important.

- There are 13 different pairs that one might draw as hole cards; all 13 appear on Sklansky's list.
- Suited hole-card holdings can be counted by considering ranks alone; if the cards are the same suit, we do not care which suit it is except as we root for more cards of that suit to appear on the board. These number

$$\binom{13}{2} = 78.$$

- Two unsuited cards of different ranks may be drawn in the same 78 ways as suited hole cards, since once again the suit is immaterial.

Adding up gives 169 possible hole-card holdings that are meaningfully different to a player.

An alternate way to reach this number is to note that each of the 13 ranks may be matched up with any other of the 13 ranks, including itself. This gives $13^2 = 169$ possibilities. Either way, Sklansky's list includes just more than half of the combinations. There may be rare occasions when betting an unlisted hand such as an unsuited 9 and 7 is justifiable—perhaps if many players have called before you and you are the last to bet—but a beginning player would do well to follow this advice [140].

Example 4.3. Suppose that your hole cards are a suited ace and king, which is the highest hole-card combination without a pair and ranks third on Sklansky's list, behind only AA and KK. Find the probability of at least one ace or king appearing on the flop.

The flop can contain 0, 1, 2, or 3 of the cards you seek. Since we're looking for "at least one" card, this question is easy to answer using the Complement Rule. We have

$$P(\text{At least one ace or king}) = 1 - P(\text{No aces or kings}).$$

The 50 cards remaining in the deck include 3 aces and 3 kings, so we have

$$P(\text{No aces or kings}) = \frac{\binom{44}{3}}{\binom{50}{3}} = \frac{473}{700} \approx .6757.$$

The corresponding probability of at least one ace or king is then .3243; close to 1 chance in 3. ■

Hole cards of consecutive ranks, suited or unsuited, are called "connectors", because they give the player a head start on making a straight. Since aces count both high and low when making straights, the number of ways to draw a pair of connectors is

$$\frac{52 \cdot 8}{2} = 208,$$

since the first card can be anything; the second card need only be one of the 8 of an adjacent rank. We divide by 2 since the order of the cards does not matter.

The probability of a pair of connectors in the hole is then

$$\frac{208}{\binom{52}{2}} = \frac{8}{51},$$

approximately 15.7%.

Example 4.4. Suppose that your hole cards are $A\heartsuit\ T\heartsuit$. Find the probability that the board completes a flush.

There are 11 hearts left unseen, and completing a flush requires 3 of them, but it doesn't matter where they fall among the flop, turn, and river. The probability of getting 3 or more hearts in 5 cards is

$$\frac{\binom{11}{3}\cdot\binom{39}{2}+\binom{11}{4}\cdot\binom{39}{1}+\binom{11}{5}}{\binom{50}{5}} = \frac{135,597}{2,118,760} \approx .0640,$$

or about 6.4%.

This includes the small probability that 5 hearts appear on the board and every player holds at least a flush, but in that circumstance, you hold the best, or *nut*, flush since the $A\heartsuit$ is in your hand. Only a straight flush can beat you if there are 5 hearts on the board.

If only one heart falls on the flop, it may be advisable to fold, but if you stay in, what is the probability of completing the flush on the turn and river?

Here, there are 10 hearts left, but you need 2 of them in 2 cards. This probability is

$$\frac{\binom{10}{2}}{\binom{47}{2}} = \frac{10\cdot 9}{47\cdot 46} \approx .0416,$$

roughly two-thirds of the probability initially computed. ■

About The Board

Since the board at Texas hold'em consists of 5 cards, many of our calculations will involve the number $\binom{52}{5} = 2,598,960$, the number of 5-card poker hands. If we have no knowledge about any player's hole cards, computations using this number as a denominator give the same result as would emerge from a more complicated calculation taking into account the possibilities for cards that have previously been dealt, as we saw on page 80.

Example 4.5. Unless the board contains at least 3 suited cards, it is impossible for any player to score a flush. What is the probability that there are 3–5 suited cards on the board?

We note that it is impossible for two players to have flushes in different suits on the same hand, since there are only 9 cards available between them. The desired probability p is found with the First Addition Rule by summing the probabilities of 3, 4, and 5 suited cards, with a multiplier of 4 to account for the different suits.

$$p = \frac{4 \cdot \left[\binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5} \right]}{\binom{52}{5}} \approx .3711.$$

■

Example 4.6. Following from Example 4.5, no player can hold a full house or 4 of a kind unless a pair occurs on the board. How likely is this?

The Complement Rule is useful here, as it's easier to calculate the probability that the board contains 5 cards of different ranks than to find the chance of a pair, 3 of a kind, or 4 of a kind, and then to combine those probabilities correctly. The chance of an unpaired board is

$$\frac{\binom{13}{5} \cdot 4^5}{\binom{52}{5}} = \frac{1,317,888}{2,598,960}.$$

The numerator in this fraction is the number of ways to draw 5 cards of different ranks; it includes the possibility that the 5 cards comprise a straight or flush and so is larger than the number of high-card hands computed in Chapter 1, which was 1,302,540.

Reducing the fraction gives

$$P(5 \text{ cards of different ranks}) = \frac{2112}{4165} \approx .5071,$$

and so the chance of at least a pair on the board is

$$1 - \frac{2112}{4165} = \frac{2053}{4165} \approx .4929,$$

just under 50%. ■

A player holding pocket aces when a third ace appears on the flop is certainly in a strong position. However, no matter what the other two cards on the board are, provided the fourth ace isn't also present and those 2 cards aren't a pair that would give our player aces full, it will always be possible for the turn to yield a straight for a player holding the right hole cards [11]. In Texas hold'em and related games, this scenario is called a *straight draw*: where a 1-card draw can complete a straight. Similarly, when a player's hand contains two suited cards and the flop contains two more cards of that suit, one can speak of a *flush draw*.

This hidden threat to a set of aces arises because the ace counts both high and low, and so both $AKQJT$ and $5432A$ are valid straights. Any card at least a 10 or less than a 5, combined with the ace that's present, opens up the chance of a straight on the turn if a player has two of the other cards of a straight with an ace. If the rest of the flop contains 2 cards between a 6 and a 9, a player holding two other cards from the 56789 or $6789T$ straights can complete that straight with the turn or the river.

Three aces, while a powerful holding worth betting, is then not quite the sure thing that it might appear to be post-flop. Indeed, any flop consisting of 3 cards of different ranks admits a straight draw, with 4 exceptions: $K72, K82, K83$, and $Q72$ [11]. The probability of a flop with a possible straight draw is then

$$\frac{\binom{52}{3} - 13 \cdot \binom{4}{2} \cdot 48 - 13 \cdot \binom{4}{3} - 4}{\binom{52}{3}} = \frac{18,300}{22,100} \approx .8281,$$

meaning that straights are possible with more than 80% of all flops. Of course, since flopping a straight draw requires certain hole cards, which may have been folded before the flop if they're a weak holding such as an unsuited $T7$, this threat may be smaller than it appears in practice.

Counting Outs and the Rule of Four

When assessing a hold'em hand after the flop, a player knows 5 cards. If that hand holds 4 cards to a flush, the probability of completing the flush on the turn is $\frac{9}{47} \approx .1915$, and the probability of missing the flush on the turn but filling it in on the river is

$$\frac{38}{47} \cdot \frac{9}{46} \approx .1582,$$

giving an overall probability of making the flush of about 34.97%.

To complete this calculation in one step, we use the Complement Rule. Suppose that the hand contains 4 clubs.

$$P(\text{Flush}) = 1 - P(\text{Draw 0 clubs}) = 1 - \frac{\binom{38}{2}}{\binom{47}{2}} \approx .3497.$$

The cards that will complete a hand are known as *outs*; here, the 4-club hand has 9 outs. More generally, if a hand has k outs, the probability P_k of catching 1 or 2 cards and completing the hand is

$$P_k = 1 - \frac{\binom{47-k}{2}}{\binom{47}{2}} = \frac{93k - k^2}{2162}.$$

This formula is too complicated to evaluate on the fly at a poker table with limited time to think, so an excellent approximation called the *Rule of Four* can be used instead [56].

Theorem 4.1. ***The Rule of Four****: The probability P_k of making a hand on the turn or river when there are k outs is approximately $4k\%$, or $\frac{4k}{100}$.*

Proof. We shall consider this proposed approximation both algebraically and graphically as evidence for its accuracy.

We seek to approximate

$$\frac{93k - k^2}{2162}.$$

Factoring in the numerator gives

$$\frac{(93-k)\cdot k}{2162}.$$

The claim of the Rule of Four is that

$$\frac{93-k}{2162} \approx \frac{4}{100} = .04.$$

The left side of this approximation is a decreasing function of k which is, to 4 decimal places, .0430 at $k = 0$ and falls to .0375 at $k = 12$. Over the interval $0 \leqslant k \leqslant 12$, then, the proposed approximation is within .0030 of .04; acceptably close to establish the desired result.

Figure 4.3 shows a graph of the difference $P_k - \frac{4k}{100}$ for $0 \leqslant k \leqslant 12$. P_k is greater than the Rule of Four estimate until the two functions are equal, at $k = 6.52$. The difference between the two functions over this interval is seen to be small, further confirming that the approximation an accurate one. If $k \leqslant 14$, the Rule of Four gives an approximation that is within 10% of P_k—and this is close enough to be useful in gameplay.

□

Table 4.1 illustrates the accuracy of the Rule of Four by comparing P_k and $\frac{4k}{100}$ numerically.

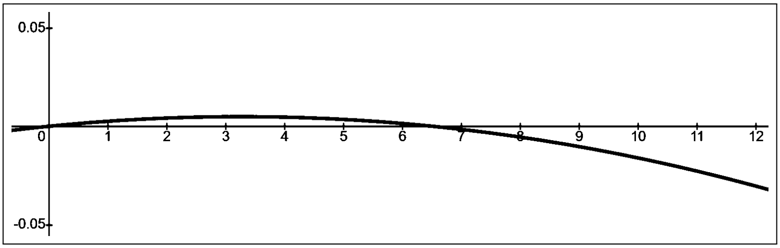

FIGURE 4.3: Accuracy of the Rule of Four: $P_k - \dfrac{4k}{100}$.

TABLE 4.1: Probability P_k of completing a hold'em hand with k outs and Rule of Four approximation ${}^{4k}\!/_{100}$.

k	P_k	${}^{4k}\!/_{100}$
1	.0426	.0400
2	.0842	.0800
3	.1249	.1200
4	.1647	.1600
5	.2035	.2000
6	.2414	.2400
7	.2785	.2800
8	.3145	.3200
9	.3487	.3600
10	.3839	.4000
11	.4172	.4400
12	.4486	.4800
13	.4810	.5200
14	.5116	.5600

Example 4.7. In hold'em, a two-way inside straight after the flop or turn, such as when a player holds $J9$ and the flop comes $KT7$, is called a *double belly-buster straight*. This hand can complete to a straight if a queen or 8 falls on the turn or river, giving 8 outs. This is twice the number of outs for a one-way inside straight, and might make a double-belly-buster straight worth betting. The Rule of Four estimates the probability of a straight as .3200, which is less than 1.8% from the true value. ∎

As the number of outs increases past 9, the accuracy of the Rule of Four drops off even as the chance of successfully catching an out rises [105]. For a better approximation when $9 \leqslant k \leqslant 18$, use

$$P_k \approx \frac{4k - (k-8)}{100} = \frac{3k+8}{100}.$$

A similar rule, suitable for estimating the probability of making a hand on a one-card draw, is the *Rule of Two*. In considering a one-card draw—the turn or the river—to a hand with k outs, this rule states that the approximate percent chance of successfully completing the hand is twice the number of outs.

For a 1-card draw with k outs from an n-card deck, the Rule of Two states that

$$\frac{\binom{k}{1}}{\binom{n}{1}} = \frac{k}{n} \approx \frac{2k}{100},$$

which is equivalent to saying that $n \approx 50$. Since the number of unknown cards on the turn is 47 and the number on the river is 46, this approximation is accurate. Multiplying by 2 is far easier than dividing by 47 or 46.

Casino Promotions

Casino poker rooms sometimes offer promotions as an incentive to players. These are games within the main game, often requiring no additional wager from players.

Bad Beat Bonus

A *bad beat* in hold'em is when a strong hand—defined, in some card rooms, to be aces full or higher—is beaten at the showdown by an even higher hand.

Example 4.8. Suppose that Robin's hole cards are $A\heartsuit\ A\clubsuit$ while Terry holds $K\diamondsuit\ Q\diamondsuit$. If the flop comes $A\spadesuit J\diamondsuit\ T\diamondsuit$, then Robin's hand is three aces and Terry holds a 4-card straight flush with a possible royal flush draw. If the turn and river are the $J\clubsuit$ and $9\diamondsuit$, Robin improves to aces full of jacks, but loses to Terry's straight flush. ■

In an effort to cushion some of the blow of a crushing disappointment when a very high hand—well into the top 1% of all 5-card hands—nonetheless falls to an opponent, some card rooms offer a Bad Beat Bonus or Bad Beat Jackpot payoff to a player whose hand is on the losing end of such a showdown.

Boulder Station, Red Rock, and Santa Fe Station, three casinos operated by Las Vegas-based Red Rock Resorts, collaborate remotely on a Jumbo Hold'em Jackpot promotion that pays a progressive jackpot on bad beats. The progressive jackpot starts at $75,000 and increases once a day by a percentage of the jackpot drop across all three participating poker rooms. As the jackpot rises, the qualifying hand falls. The $75,000 starting value requires that a hand of 4 queens or higher lose in the showdown, while a jackpot of $350,000 or higher can be won by a losing $AAATT$ or a better hand.

The progressive jackpot is divided as follows:

- 15% goes to the holder of the beaten hand. If 2 or more high hands lose on the same hand, this share is awarded only to the highest-ranked losing hand.

- 10% is awarded to the winner of that hand.
- 5% is divided evenly among all of the players at the table where the jackpot was won, including the winner and loser.
- The remaining 70% is divided among all active players at the 3 participating poker rooms when the jackpot was hit, including the players at the decisive table who shared 5% of the jackpot earlier.

Certain restrictions are placed on qualifying bad-beat hands. All four of a kind hands require a pocket pair, which reduces the potential field of winners to 1 player. If 4 of a kind shows on the board, an event with probability $p = {}^1\!/_{4165}$, the hand is won by the player with the highest kicker, since no straight or royal flushes are possible, but this does not win a share of the jackpot. Four of a kind plus an ace on the board creates a tie among all players still in the hand.

All other qualifying hands must be made using both of the player's hole cards to be eligible.

One scenario that often leads to bad beats is when 3 aces fall on the flop; this has probability

$$\frac{\binom{4}{3}}{\binom{52}{3}} = \frac{1}{5525}.$$

Any player who played a pair in the hole now has aces full, but they will lose to a player who holds the fourth ace among their hole cards and cannot catch up by drawing into a straight flush, since they hold cards of 2 ranks with only the turn and river to come.

The probability that the board contains exactly 3 aces, which might trigger a bad beat bonus even if the aces aren't all part of the flop, is

$$\frac{\binom{4}{3} \cdot \binom{48}{2}}{\binom{52}{5}} \approx \frac{1}{576}.$$

In the Red Rock Resorts promotion described above, a player whose pair in the hole is completed to 4 of a kind by the turn and river could claim the loser's share of the prize pool by losing to 4 aces—provided that his or her 4 of a kind met the minimum hand requirement. Only when the jackpot rises to $300,000 do all beaten 4 of a kinds qualify for the jackpot.

Example 4.9. Jackpots of this size are only awarded when low-probability events happen. How low is "low"?

Suppose you are playing in a 6-handed game. If your hole cards are 9♣ 9♡ and 3 aces come on the flop ($p = {}^1\!/_{5525}$; above), the probability that the turn and the river are then the 9♠ and 9♢, in either order, is

$$\frac{1}{\binom{47}{2}} = \frac{1}{1081}.$$

If this happens, the probability that one of your 5 opponents holds the fourth ace and beats your four 9s with four aces is

$$\frac{\binom{44}{9}}{\binom{45}{10}} = \frac{2}{9}.$$

If this sequence of 3 improbable events happens *and* the jackpot is at least $160,000, you qualify for the loser's share of the jackpot. The exact amount of your win depends on how many players are at the 3 poker rooms involved, but it'll be at least $25,333.33. ■

Example 4.10. At the Mandalay Bay poker room on the Las Vegas Strip, the Aces Cracked promotion in the fall of 2021 offered a $100 consolation prize to any player whose pocket aces wound up as a losing hand in a pot amounting to at least $10. This prize was awarded up to 4 times during each gaming session, which ran from 3:00–6:00 P.M. Thursday through Monday.

A payoff of this nature invites collusion among players, since the prize for winning a $30 pot by cracking an opponent's aces is far less than the $100 jackpot paid to the losing player. Players in such a game might share information about their hands and agree to split the proceeds of the bonus, which is contrary to the "one player, one hand" rule that is standard in card rooms. The official rules of the promotion explicitly stated that any player discussion of hands during play forfeited eligibility for this jackpot. ■

Bad beat jackpots are frequently funded through additional rakes from each pot in addition to the standard per-hand rake. The Mandalay Bay Aces Cracked jackpot described in Example 4.10 drew $1 from each pot of $10 or more and an additional dollar from every pot that reached $40. This represents an additional rake of 10% from a $10 pot, decreasing to 2.56% of a $39 pot and then jumping to 5% when the pot hits $40. The jackpot rake percentage then decreases as the pot rises, since the jackpot rake is fixed at a $2 maximum per hand and no further deductions are taken.

First Five

At the Orleans Casino in Las Vegas, the First Five bonus has been offered 2 days a week. This bonus opportunity is paid to gamblers at Texas hold'em tables who achieve certain hands among their hole cards and the flop. Table 4.2 shows the pay table. In order to claim the First Five bonus, a player must

TABLE 4.2: First Five payoffs, Orleans Casino

Winning hand	**Payoff**
Ace-high flush	\$50
Aces full	\$75
Ace-high straight	\$100

hold the indicated hand among the first 5 cards and must have at least one ace among his hole cards. A royal flush in 5 cards receives only the \$100 bonus for an ace-high straight, not the ace-high flush bonus.

As a free promotion, the expectation of First Five is positive; as a 5-card hand promotion, it can be analyzed without the need to consider the turn and river. What value does First Five add to each hand?

The probability of each of the 3 hands is easily calculated, but we must reduce these numbers by an amount that addresses the requirement that there be at least one ace in the hole. For the flush and straight, which involve only 1 ace, the probability that the ace is in the hole is $\frac{2}{5}$; for aces full, we have

$$P(\text{Ace in the hole}) = 1 - P(\text{No aces}) = 1 - \frac{\binom{3}{2}}{\binom{5}{2}} = 1 - \frac{3}{10} = \frac{7}{10}.$$

We then have the following probabilities for the 3 First Five bonus hands:

$$P(\text{Ace-high flush}) = \frac{4 \cdot \binom{12}{4} - 4}{\binom{52}{5}} \cdot \frac{2}{5} \approx 3.0412 \times 10^{-4}$$

$$P(\text{Aces full}) = \frac{\binom{4}{3} \cdot 12 \binom{4}{2}}{\binom{52}{5}} \cdot \frac{7}{10} \approx 7.7569 \times 10^{-5}$$

$$P(\text{Ace-high straight}) = \frac{4^5}{\binom{52}{5}} \cdot \frac{2}{5} \approx 1.5760 \times 10^{-4}.$$

The expected value of First Five is

$$E = (50) \cdot 3.0412 \times 10^{-4} + (75) \cdot 7.7569 \times 10^{-5} + (100) \cdot 1.5760 \times 10^{-4} \approx \$0.037,$$

approximately 3.7¢. If you can sell your interest in First Five to a fellow player for 4¢ per hand, you're getting more than this promotion is worth.

4.2 Different Decks

Short Deck

Deck composition:	36 cards
Hand size:	7 cards
Community cards:	5
Betting rounds:	4

Short Deck, also known as *Six Plus*, is a variation of Texas hold'em that uses a standard deck with the 2s through 5s removed, leaving 36 cards. The play of the game with the flop, turn, and river is unchanged. Short Deck originated in Asia several years before making its World Series of Poker debut in 2019 [73].

The inventors of Short Deck, Paul Phua and Richard Yong, set out to create a game that allowed amateurs who prefer lots of action to play on more equal footing with professionals who are more cautious. This is achieved in Short Deck by introducing a game that has not been so completely solved by players and mathematicians [65]. The diminished deck calls for different playing strategies, and a skilled hold'em player who cannot make these adjustments will be at a disadvantage against a knowledgeable Short Deck player.

As in hold'em, an ace can be used as either the high or low card in a Short Deck straight, so the hand 9876A is a straight, one which might be overlooked by some players [104]. This means that an ace together with any other card in the hole can draw out to a straight when the board cards are revealed, but that may not be an argument in favor of always calling with Ax in the hole. Some Short Deck players frequently remind themselves that "an ace is a five" when playing, to keep this important idea at the front of their minds.

The net effect of the diminished deck is to make the higher hands more likely. We saw above that the probability of pocket aces in Texas hold'em is $\frac{1}{221}$. In Short Deck, this probability rises, to

$$\frac{\binom{4}{2}}{\binom{36}{2}} = \frac{1}{105} \approx .0095$$

—still low, but more than double the previous value. At the same time, there is no real analog to the unsuited 2–7 as the worst possible pair of hole cards, since the lowest 2-card hand without a straight or flush draw is $6J$ unsuited, which contains a relatively high card in the jack. (This pair of cards requires 4 additional cards to promote to a straight, so it is not regarded as admitting a straight draw.) The probability of an unsuited $J6$ in the hole is

$$\frac{4 \cdot 3}{\binom{36}{2}} = \frac{2}{105} \approx .0190.$$

Once again, the probability of pocket aces is half the probability of the lowest possible starting cards.

The best hole cards for pursuing a straight are JT. Both JT and $T9$ can draw out to 4 of the 6 possible Short Deck straights, but the straights containing JT collectively rank higher than those containing $T9$. Either holding can lead to $789TJ, 89TJQ$, and $9TJQK$, but JT can also produce $TJQKA$ with 3 additional cards, which beats the $6789T$ that is also possible with $T9$ in the hole. If your hole cards are an unsuited JT, the probability that the flop completes a 5-card straight is

$$\frac{4 \cdot 4^3}{\binom{34}{3}} = \frac{8}{187} \approx .0427,$$

which is more than 3 times greater than the probability of flopping a straight with the same hole cards in hold'em. This happens because the low cards in the deck are barriers to a straight built around JT when they appear on the flop, and many of those cards have been removed from the deck [65].

An added advantage to starting with JT over $T9$ is the pursuit of a straight is that every straight starting with JT is the nut straight, which cannot be beaten. Completing a $T9$ holding to $KQJT9$ with a king, queen, and jack on the board leaves a straight that can be beaten by an opponent holding AT in the hole.

Example 4.11. However, eliminating low cards means that there are fewer cards with which to build flushes. Considering 5-card hands, show that a flush is less likely than a full house in Short Deck, which is the reversal of their probabilities in a full deck.

There are 9 cards in each suit, so the number of 5-card flushes is

$$4 \cdot \binom{9}{5} - 20 = 484,$$

where 20 straight flushes and royal flushes are removed. Full houses number

$$9 \cdot \binom{4}{3} \cdot 8 \cdot \binom{4}{2} = 1728.$$

■

Example 4.11 suggests taking a look at the standard poker hand rankings to see what other changes in the order occur when moving to Short Deck.

Table 4.3 shows how the hands rank based purely on their Short Deck probabilities. Hands that rank higher at Short Deck than in standard Texas hold'em are italicized. As one can see, the relative rank of a straight decreases in the move to a 36-card deck.

TABLE 4.3: Short Deck: Hands ranked by probability [65].

Royal flush
Straight flush
Four of a kind
Flush
High card
Full house
Three of a kind
Straight
Two pairs
Pair

Example 4.12. There are $\binom{36}{7} = 8,347,680$ ways to choose the 7 cards that comprise a Short Deck hand. How many of these contain 7 cards of different ranks?

Once the ranks have been chosen, which can be done in $\binom{9}{7} = 36$ ways, there are 4 choices for each card. It follows that $36 \cdot 4^7 = 589,824$ Short Deck hands contain no pair. The probability of a no-pair hand is about 7.07%. ■

Example 4.13. Of the hands identified in Example 4.12, how many do *not* contain a straight?

There are 36 ways to choose the ranks. This question may be answered by appealing to the symmetry of the formula for combinations:

$$\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!} = \frac{n!}{(n-r)! \cdot [n-(n-r)]!} = \binom{n}{n-r},$$

which states that the number of ways to choose r items from a set of n is the same as the number of ways to choose the $n-r$ items that are left behind. There are therefore 36 ways to choose the 2 ranks that are not represented in the 7 cards in hand. Examining this list of combinations shows that 15 leave no straights behind. Two key cards for breaking up 5-card straights in Short Deck are the 9 and T, which sit in the middle of the depleted deck.

Multiplying by the number of choices for suits gives $15 \cdot 4^7 = 245,760$ hands with 7 ranks and no straight. ■

Referring to Table 4.3, ranking a no-pair high card hand in its proper place would be too disruptive to game play, so Short Deck continues to rank that hand at the bottom. Opinions differ on how to handle the other deviations from standard hand ranks. A version of Short Deck called *Triton Poker* calls for straights to outrank three of a kind, as is the case in ordinary poker games. At the World Series of Poker, the game is called Short Deck, but the Triton Poker rankings are used. A more conventional version respects the mathematics, following the probabilities with three of a kind beating a straight [65]. In both versions, flushes beat full houses. If 3 of a kind beats a straight, the advantages of JT over $T9$ detailed above are somewhat diminished.

Example 4.14. The hole cards $K\diamondsuit Q\diamondsuit$ together with the board $K\clubsuit\ K\spadesuit\ T\heartsuit\ 9\clubsuit\ J\diamondsuit$ can be played as either 3 kings or a king-high straight, depending on the hand rankings in use. ■

With a smaller deck, Short Deck players can replace the Rule of Four and the Rule of Two by new corresponding Rules of Six and Three in making strategy decisions [66]. We can confirm the Rule of Six by following the derivation on page 116. The probability of hitting one of the k outs possessed by a given Short Deck hand after the flop is

$$P_k = 1 - \frac{\binom{31-k}{2}}{\binom{31}{2}} = \frac{61k - k^2}{930}.$$

That this is approximately equal to $\frac{6k}{100}$ can be seen by graphing the difference $P_k - .06k$. Figure 4.4 shows this difference, which is less than 1.5% if $k \leqslant 7$.

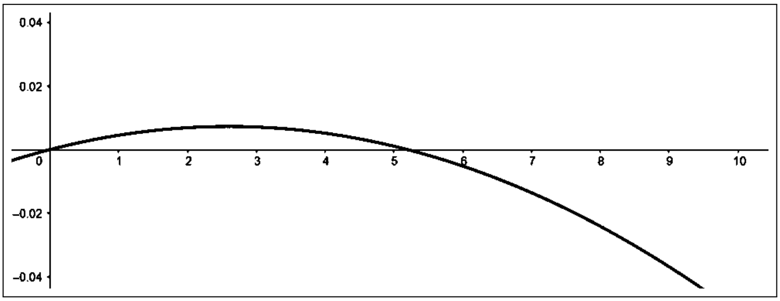

FIGURE 4.4: Accuracy of the Rule of Six: $P_k - \frac{6k}{100}$.

As a further check, Table 4.4 compares P_k and $.06k$ numerically.

This makes the Rule of Six an entirely acceptable approximation. Confirmation of the Rule of Three is left as an exercise. The smaller deck allows for an increase in the multiplier. Of course, everyone at the table is also multiplying their outs by these higher coefficients.

TABLE 4.4: Probability P_k of completing a Short Deck hand with k outs and Rule of Six approximation $6k/100$.

k	P_k	$6k/100$
1	.0645	.0600
2	.1269	.1200
3	.1871	.1800
4	.2452	.2400
5	.3011	.3000
6	.3548	.3600
7	.4065	.4200

When considering the best possible hole cards in Short Deck, one naturally regards AA and KK as having the same strength that they have in Texas hold'em. QQ also holds considerable value, but the next best starting hand, and the first not to contain a pair, is JT, either suited or unsuited [67]. JT outranks AK, the first non-pair hand among the hole card rankings for hold'em, because of its ability to block opponent straights and because the straights it can draw into are invariably the nut straight.

The probability of a suited JT in the hole is simply

$$\frac{4}{\binom{36}{2}} = \frac{4}{630} \approx .0063;$$

the probability of an unsuited JT is 3 times greater, about .0190.

California Hold'em

Deck composition:	60 cards
Hand size:	7 cards
Community cards:	5
Betting rounds:	4

In the midst of the American poker boom in the 2000s decade, some players and game designers were casting about for the next big game: one that might capture players' attention as Texas hold'em had. One game offered as a candidate used a nonstandard deck with more than 52 cards: *California hold'em* was played with a 60-card deck. Any game involving playing cards represents some sort of balance between skill and luck; this game was designed in part as a reaction to poker developments in the early 21st century that seemed to be tilting that balance unacceptably far toward luck. The deck added four 11s and 12s to the deck, which ranked in numerical order between the 10 and jack [113].

So far, so good. Part of the game designer's rationale for creating California hold'em was to increase the role of skill in hold'em. The player with superior cards, it was said, should be able "to price poor players out of hands that they should not be involved with to begin with" [3]. This preference for skill stands in direct opposition to Short Deck and its designers' interest in building a game where experts and novices might be on more equal footing.

There were some twists to the rules that made the game a bit more intricate [3]:

- A royal flush was defined, reasonably enough, as a suited $AKQJ12$.
- The 10s and 11s were unsuited cards, so were not available for flushes.
- *Both* hole cards had to be used for a straight or flush, including straight flushes. Other hands could use only 1 hole card, or could play the board.

  An implication of this third point is that a royal flush on the board is no royal flush; moreover, a royal flush according to these rules was an unbeatable hand. The number of 7-card hands containing a royal flush drawn from a California hold'em deck is

$$4 \cdot \binom{55}{2} = 5940,$$

  but the fraction of these in which the player's hole cards are both part of the royal flush is

$$\frac{\binom{5}{2}}{\binom{7}{2}} = \frac{10}{21},$$

  less than 50%.

While the rules for making straights and flushes in California hold'em might complicate the process of counting outs after the flop, estimating the chance of catching an out on the turn or river is a simple matter once the number of outs is determined. The chance of catching 1 or 2 of k outs among the last 2 board cards is

$$1 - \frac{\binom{55-k}{2}}{\binom{55}{2}} = \frac{109k - k^2}{2970}.$$

Numerical experimentation shows that multiplying the number of outs by 3.5 gives a good approximation to the percentage probability of succeeding, that is:

$$\frac{109k - k^2}{2970} \approx \frac{3.5k}{100},$$

so California hold'em players can adopt the Rule of 3½ to estimate their chance of hitting one of their outs—provided that they are proficient in multiplying by fractions or decimal numbers.

Example 4.15. Suppose that the flop and turn in a hand of Texas hold'em consist of

$$9\clubsuit\ J\spadesuit\ 7\clubsuit\ 2\clubsuit.$$

If Alex holds $J\heartsuit\ J\diamondsuit$ while Sandy's hole cards are $8\clubsuit\ 3\heartsuit$, Alex holds a huge advantage [3]. However, Sandy can see both a flush draw and an inside straight draw and so might stay in the hand longer than is mathematically wise—indeed, an experienced hold'em player would almost surely have folded $8\clubsuit\ 3\heartsuit$ prior to the flop. The chance—however small—that low-skill decisions might be rewarded was part of the motivation for California hold'em.

In Texas hold'em, Sandy has 11 outs (8 clubs, not including the $J\clubsuit$, and 3 other 10s) and so has a 25% chance of winning on the river despite having decidedly inferior hole cards. In California hold'em, Alex's three of a kind cannot lose on the river since no flush or straight can include the unsuited 8 and 3 in Sandy's hand. ■

The diminution of skill becomes more acute when several poor players make long-shot bets against a single good player.

Royal Hold'em

Deck composition:	20 cards
Hand size:	7 cards
Community cards:	5
Betting rounds:	4

Royal hold'em reduces the deck even more than Short Deck does, to 20 cards: the 10s through aces in 4 suits. Since a 5-card hand is constructed from 7 cards with only 5 ranks, it is not possible to have a one-pair or high card hand, and so the lowest possible hand is two pairs. Any 5-card no-pair hand dealt from this 20-card deck must be a straight. Moreover, with only 5 cards per suit, the only flushes possible are royal flushes.

As in Short Deck, a shrunken deck changes the ranks of some hands. There are $\binom{20}{7} = 77,520$ possible 7-card hands in royal hold'em. Reducing these to playable 5-card hands calls for careful consideration of 7-card poker hands.

In effect, royal hold'em is a game played with 7-card hands drawn from a 20-card deck. This changes both the number and type of possible hands. Royal hold'em hands are ranked in the order shown in Table 4.5.

The lowest hand in royal hold'em is 2 pairs; this only arises when the 7 cards contain 3 pairs. A 2-pair hand must have the 7-card configuration *xxyyzza* with no two letters representing the same rank. A hand of the form

TABLE 4.5: Royal hold'em: Hands ranked by probability.

Royal flush
Four of a kind
Full house
Straight
Three of a kind
Two pairs

xxyyabc will be played as a straight or a royal flush (remember that there are only 5 ranks), and *xxyyzzz* will play as a full house. This means that no final 2-pair hands will contain a pair of 10s, since the 7-card hand will always contain 2 higher pairs.

Counting *xxyyzza* hands gives

$$\binom{5}{3} \cdot \binom{4}{2}^3 \cdot 8 = 17,280$$

2-pair hands at the showdown.

At the opposite end of Table 4.5, the 4 royal flushes in a deck can each be filled out to a 7-card hand with

$$\binom{15}{2} = 105$$

combinations of 2 other cards. As a result, the probability of a royal flush is

$$\frac{4 \cdot 105}{77,520} = \frac{420}{77,520} \approx .0054,$$

approximately once in every 185 hands. In 4 of these 420 hands, the royal will be on the board, and all players who stay in through the end will split the pot.

The overall probability of a royal flush on the board is simply

$$\frac{4}{\binom{20}{5}} = \frac{1}{3876}.$$

4.3 Omaha

Deck composition:	52 cards
Hand size:	9 cards
Community cards:	5
Betting rounds:	4

Omaha is a variant of hold'em that was called Nugget Hold'em when it debuted in Las Vegas at the downtown Golden Nugget Casino in the early 1980s [88]. There is apparently no intentional connection between the game and Nebraska's largest city; inventor Robert Turner has commented that when the game was introduced as Nugget Hold'em, a frequent early patron was from Omaha, which may have led to the name which has endured [145].

Omaha gameplay is much like Texas hold'em, but the rules for assembling a hand differ. Players are dealt 4 hole cards instead of 2 and must use exactly 2 of them to build their final hand. A player dealt all 4 aces as hole cards can only use 2 of them—and will not see any help in the form of additional aces among the 5 community cards. Indeed, four of a kind is among the least desirable sets of Omaha hole cards due to its limited potential to improve when the board is dealt, even as a draw or stud poker player would rejoice at such a holding.

Example 4.16. Consider the 2 Omaha hands shown in Figure 4.5, between Lou on the left and Robin on the right.

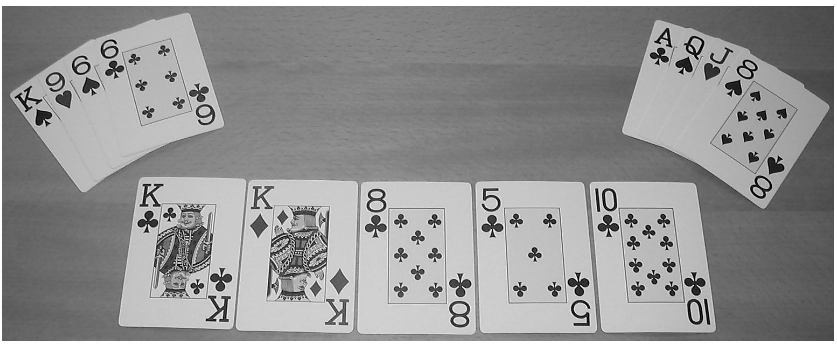

FIGURE 4.5: A 2-player Omaha showdown [103].

If not for the restriction on how many hole cards may be used in the final 5-card hand, Lou would win with kings full of 6s, which would beat Robin's ace-high flush in clubs. Since only 2 hole cards may be used in the final Omaha hand, Lou can only reach 3 kings. Robin misses out on both a club flush and an ace-high straight, and must play kings over 8s, which loses. ■

In casinos and card rooms, Omaha is often played as "Pot-Limit Omaha" or PLO, and raises are limited to the size of the pot. By contrast with no-limit hold-em games, players cannot go all in and bet all of their chips. This betting restriction works to decrease variance of players' bankrolls and so makes the game a bit less volatile.

With 4 hole cards from which to choose 2, each player has $\binom{4}{2} = 6$ possible choices of hole cards, increasing the chance that a dealt holding will be worth backing through the flop. After the board is complete, there are then $\binom{5}{3} = 10$ ways to choose 3 community cards, so an Omaha hand played through to the showdown can be used to make 60 different 5-card hands. By contrast, a Texas hold'em hand admits only $\binom{7}{5} = 21$ different hand choices.

While identifying the best possible Omaha hole cards is challenging due to the multiple possibilities that exist, one particularly attractive feature is when the 4 cards include only 2 suits, which opens up 2 different flush draws. Such a holding is called *double-suited*. Suppose that your hole cards are $A\heartsuit\ Q\spadesuit\ 8\heartsuit\ 7\spadesuit$ and that the flop comes $J\spadesuit\ 6\heartsuit\ 4\clubsuit$. The chance of the turn and river completing a 5-card flush for you is

$$\frac{2 \cdot \binom{10}{2}}{\binom{45}{2}} = \frac{1}{11} \approx .0909,$$

just more than 9%. With a suited pair of hole cards in Texas hold'em and a third card of that suit among the flop, the chance of completing a flush is only

$$\frac{\binom{10}{2}}{\binom{47}{2}} \approx .0416,$$

less than half the value for Omaha.

The probability of a double-suited set of hole cards with 2 cards of each suit is

$$\frac{\binom{4}{2} \cdot \binom{13}{2}^2}{\binom{52}{4}} \approx .1348.$$

This is not an extremely rare event.

Example 4.17. Suppose that your hole cards form two pairs, such as $TT88$. Find the probability that the flop gives you 3 of a kind. This is called "flopping a set".

Among the 3 cards of the flop, we want to find either a 10 or an 8; there are 4 of those cards remaining. If we confine our attention to flops that yield a single set, the probability is

$$\frac{\binom{4}{1} \cdot \binom{44}{2}}{\binom{48}{3}} = \frac{473}{2162} \approx .2188.$$

The probability of flopping two sets without flopping 4 of a kind is

$$\frac{2^2 \cdot 44}{\binom{48}{3}} \approx .0102,$$

and the probability of flopping 4 of a kind is

$$\frac{2 \cdot 46}{\binom{48}{3}} = \frac{1}{188};$$

this includes the extremely rare cases where the flop is either $TT8$ or $T88$ and 4 of a kind is joined by a set. Adding up all 3 probabilities gives

$$P(\text{Flop a set or better}) = \frac{1013}{4324} \approx .2343$$

—nearly 1 chance in 4. ■

The probability that the flop contains exactly a pair (not 3 of a kind) in Omaha is

$$\frac{13 \cdot \binom{4}{2} \cdot 48}{\binom{52}{3}} \approx .1694.$$

When this happens, a difference between Omaha and Texas hold'em emerges. In hold'em, a pair on the board has a low chance of giving someone a full house, since there are only 2 hole cards per player. If the flop is $JJ3$, only a pair of 3s or $J3$ gives an immediate full house.

In Omaha, with 4 hole cards per gambler, the chance that someone will make a full house goes up considerably with a pair on the board [12]. With $JJ3$ as the flop, a player holding $J3$ among their 4 hole cards is said to have the *overfull* house, while a player with 33 has the *underfull* house. The overfull house is considerably more valuable. As the board is completed, players who hold connected small cards in the hopes of drawing into a straight may find their hands leading to higher full houses. If a 5 appears on the turn, a player with 4556 in the hole now holds 5s over jacks, which beats 3s over jacks but loses to the overfull house of jacks over 3s.

Omaha High-Low

With such variety possible in the player's hands, Omaha is sometimes played as a high-low game where players may choose different cards to form their best high and low hands. The low hand may be required to be an 8-high or lower hand, and the game is then called "Omaha 8 or better", or simply Omaha-8. Straights and flushes do not count when determining the low hand.

Example 4.18. Suppose that the game in Example 4.16 is played high-low with straights and flushes not counted against a low hand. Lou's best low hand is the 10-high hand $T\clubsuit$ $9\heartsuit$ $8\clubsuit$ $6\clubsuit$ $5\clubsuit$. Robin can use the $A\clubsuit$ as a low card with the $J\heartsuit$ to avoid pairing the $8\clubsuit$ on the board, but can only make the jack-high hand $J\heartsuit$ $T\clubsuit$ $8\clubsuit$ $5\clubsuit$ $A\clubsuit$.

If the game is Omaha-8, neither player has a qualifying low hand, and Lou takes the entire pot with the better high hand. If there is no restriction on the low hand, Lou wins both high and low, scooping the pot. ■

How many qualifying low hands are possible? Since straights and flushes do not disqualify a low hand, there are $\binom{8}{5} = 56$ ways to choose the ranks of the final 5-card hand. The suits may then be selected in $4^5 = 1024$ ways; multiplying gives a total of 57,344 possible hands. This is a mere 2.21% of all 5-card hands, which explains in part why it's not uncommon for a round of Omaha-8 to conclude with no qualifying low hand.

Example 4.18 illustrates the fact that if an Omaha-8 board does not contain at least 3 cards of different ranks of 8 or less, including aces ("low cards"), no player can build a qualifying low hand. What is the probability that this happens?

This question can be answered with the Complement Rule and consideration of some subcases. We have

$$P(\text{Board lacks at least 3 low cards}) = 1 - P(\text{Board has 3–5 low cards}).$$

The number of ways that the board can contain exactly 3 low cards of different ranks is

$$\binom{8}{3} \cdot 4^3 \cdot \binom{20}{2} = 680,960,$$

where the third factor above gives the number of ways to choose 2 high cards from the 20 cards, 9 through king, in the deck.

A 5-card board with 4 low cards can look like $xxyzH$ or $xyzwH$, where each lower-case letter represents a different low rank and the H stands for a high card. The number of $xxyzH$ hands is

$$\binom{8}{1} \cdot \binom{4}{2} \cdot \binom{7}{2} \cdot 4^2 \cdot \binom{20}{1} = 322,560$$

and the number of $xyzwH$ hands is

$$\binom{8}{4} \cdot 4^4 \cdot \binom{20}{1} = 358,400.$$

Taken together, there are 680,960 boards with exactly 4 low ranks present.

Finally, the number of boards with 5 low cards can be counted by starting with $\binom{32}{5} = 201,376$, the number of ways to choose 5 low cards, and subtracting the boards of the form $xxxyy$ and $xxxxy$ which don't have 3 different ranks present. This number is

$$\binom{32}{5} - \binom{8}{1} \cdot \binom{4}{3} \cdot \binom{7}{1} \cdot \binom{4}{2} - \binom{8}{1} \cdot \binom{4}{4} \cdot 28 = 199,808.$$

We then have

$$P(\text{Board has 3–5 low cards}) = \frac{680,960 + 680,960 + 199,808}{\binom{52}{5}} = \frac{664}{1105},$$

and so

$$P(\text{Board lacks at least 3 low cards}) = 1 - \frac{664}{1105} = \frac{441}{1105} \approx .3991.$$

Just under 40% of Omaha-8 boards will make a qualifying low hand impossible for all players.

The best hole cards for Omaha, assuming that this game is being played for high, are $AAKK$ of 2 suits, as $A\heartsuit\ A\clubsuit\ K\heartsuit\ K\clubsuit$, which sets a player up for high three-of-a-kinds and nut straights and flushes. For high-low Omaha, $AA23$ in 2 suits is the best holding, as it gives good cards to pursue both high and low hands.

Example 4.19. Scooping the pot in Omaha-8 is much easier when you hold an ace among your hole cards. What is the probability of at least 1 ace in the hole?

Once again, we use the Complement Rule. $P(\text{At least 1 ace}) = 1 - P(\text{No aces})$, and we then have

$$1 - P(\text{No aces}) = 1 - \frac{\binom{48}{4}}{\binom{52}{4}} \approx .2813.$$

This includes the very small probability that you hold all 4 aces, which makes you hand worthless in competing for low. This probability is

$$\frac{1}{\binom{52}{4}} = \frac{1}{270,275},$$

which can be ignored without significant error. ■

A looming threat when playing Omaha-8 is *counterfeiting*, which happens when a low hole card is paired on the board, thus making a winning low hand harder to achieve. Suppose, for example, that your hole cards are $A\heartsuit\ 2\diamondsuit\ 9\spadesuit\ T\clubsuit$. The probability of your hand being counterfeited on the flop by an ace or deuce is

$$1 - \frac{\binom{42}{3}}{\binom{48}{3}} \approx .3363,$$

roughly 1/3 of the time.

Big O

Deck composition:	52 cards
Hand size:	9 cards
Community cards:	5
Betting rounds:	4

Big O is a variation on high-low Omaha in which players are dealt five hole cards instead of four. Players must make their best hand from exactly two of their hole cards and three board cards, giving

$$\binom{5}{2} \cdot \binom{5}{3} = 100$$

possible hands from a player's 10 cards. Once the showdown is complete, players show their cards; the players with the highest and lowest hands split the pot. If no player has a qualifying low hand of an 8-high or lower, the player with the highest hand wins the entire pot.

Five Card Double Board Omaha 8 Or Better Ultimate

Deck composition:	52 cards
Hand size:	10 cards
Community cards:	10
Betting rounds:	4

Among the roster of poker variants at Resorts World's Coach's Game is *Five Card Double Board Omaha 8/OB Ultimate* [68]. "Five Card" refers to the number of hole cards dealt to each player; "Double board" means that 2 sets of community cards are dealt, simultaneously. The pot is then split between the player with the best high hand and best 8-high or lower low hand using 2 of their hole cards and the cards from either board.

Example 4.20. Suppose that the two boards are these:

$$\mathbf{A} : A\diamondsuit\ K\diamondsuit\ J\clubsuit\ T\spadesuit\ 5\clubsuit$$
$$\mathbf{B} : 8\spadesuit\ 6\clubsuit\ 4\clubsuit\ 3\spadesuit\ 3\heartsuit$$

We note that no flushes are possible, since neither board contains 3 cards of a single suit.

- Terry holds $T\diamondsuit$ 8$\diamondsuit$ 7$\spadesuit$ 5$\heartsuit$ 4$\diamondsuit$ and can make the straight 87654 or the slightly lower low hand 87643 with board B. The straight beats the two pair hand, 10s and 5s, that can be made with board A.
- Chris's hole cards are $A\heartsuit$ $K\heartsuit$ 9$\spadesuit$ 7$\clubsuit$ 6$\diamondsuit$, enabling the low hand 8643A on board B. This straight uses the 6$\diamondsuit$ in Chris's hand, since the final hand must use 2 hole cards. The best possible high hand is aces and kings, using board A.
- Pat chooses board A to go with $Q\spadesuit$ $Q\diamondsuit$ $J\heartsuit$ $T\heartsuit$ 2$\heartsuit$, forming a Broadway straight. Pat's best low hand is T6432, using board B, but this is too high to qualify as 8-or-better.
- Sandy's $A\clubsuit$ $J\spadesuit$ 9$\clubsuit$ 5$\spadesuit$ 2$\spadesuit$ is good for a low 8432A using board B and aces and jacks as high with board A.
- Alex holds $Q\clubsuit$ 8$\clubsuit$ 3$\diamondsuit$ 2$\diamondsuit$ 2$\clubsuit$ and uses board B to make a full house of 3s over 8s, which is probably stronger than the low 86432 that's possible as a low hand on board B.

Alex wins with the top high hand, while Sandy edges out Chris for the best 8-high low hand.

■

Double board poker is related to "running it twice", a game variation in Texas hold'em and similar community-card games where two or more players are all-in and agree that two separate boards will be completed. If the players are all-in after the flop, this translates into two distinct turn and river combinations. The pot is split between the winners on each board; a player can win the entire pot by winning on both boards. In the long run, running a board twice cuts down on an individual player's variance while the expectation remains the same (page 143).

Since each player can make 100 hands on each board, for a total of 200 hands, it is quite possible that at least one of the hands on one of the boards will be a contender for high or low hand. The sheer number of possibilities makes it likely that several players will remain in the betting until the final showdown. The excitement accompanying bigger pots is somewhat tempered by the fact that the final pot is split 2 ways—thought scooping the pot with a low straight, among other possibilities, can still happen.

Drawmaha

Deck composition:	52 cards
Hand size:	10 cards
Community cards:	5
Betting rounds:	4

Drawmaha, alternately spelled *Dramaha*, is an umbrella term for a collection of Omaha/draw poker hybrids. In the most basic version of Drawmaha, players are each dealt 5 hole cards, and the pot is split between the best Omaha high hand and the best 5-card hand—however that may be defined—in a player's hole cards. Basic Drawmaha awards half the pot to the best 5-card draw hand. Players are given the opportunity to discard from their hole cards and draw new ones after the second round of betting, following the flop.

Drawmaha somewhat mitigates the effect of holding hole cards that are "too good" when taken as a set, since a good 5-card hand can contend for half the pot in its own right, independent of how 2 cards might fit with the board to form a different 5-card hand.

Drawmaha 49

Deck composition:	52 cards
Hand size:	10 cards
Community cards:	5
Betting rounds:	4

In *Drawmaha 49*, half the pot goes to the player with the highest-scoring hole cards, using a numerical evaluation where face cards each count 0 and other cards count their pip value from 1 for aces to 10 for tens. The best possible hand, four 10s and a 9, scores 49 points and gives the game its name.

Novice Drawmaha 49 players with blackjack experience may make the error of assuming that 10s and face cards score the same, which they do not. Whether by assuming that face cards score 10 or that 10s score 0, this can lead to some unwise betting based on misread hole-card hands.

With this scoring system in use, an average card has value $\frac{55}{13} \approx 4.23$, so an average 5-card hand counts about 21 points.

Drawmaha Zero

Deck composition:	52 cards
Hand size:	10 cards
Community cards:	5
Betting rounds:	4

Drawmaha Zero modifies Drawmaha 49 by giving half the pot to the lowest-scoring hole cards using Drawmaha 49 accounting. Five face cards is the best holding, totaling 0 points. As we saw earlier in counting blazes, there are

$$\binom{12}{5} = 792$$

ways to draw a 0-point Drawmaha Zero hand.

4.4 Pineapple Poker

Deck composition:	52 cards
Hand size:	8 cards
Community cards:	5
Betting rounds:	4

Pineapple Poker is a hold'em variant that is said to have been invented in Hawaii—a state with no legal gambling—and brought to the Golden Nugget Casino in downtown Las Vegas by a visitor from the islands [5]. In this game, players are initially dealt 3 hole cards instead of 2, and must discard one before the flop is dealt. The timing of the discard separates pineapple from Omaha and its variants: one hole card must be permanently removed from play before the community cards are revealed. This gives the player 3 options for a pair of hole cards, which encourages more players to stay in at least through the flop and leads to higher-ranked winning hands relative to Texas hold'em.

The opportunity to discard one card may backfire on the player, of course. A player dealt three of a kind must discard one of the trio, which eliminates the chance of drawing into 4 of a kind and greatly diminishes the chance of restoring a three-of-a-kind hand. For other hands, though, the knowledge that one card is out of play may be useful as the board is revealed.

Example 4.21. A player dealt $K\diamondsuit$ $K\clubsuit$ $A\spadesuit$ could discard the ace and know that not only does he hold pocket kings, but that one of the aces is out of the deck and cannot be drawn by any opponents hoping to complete their hands. Discarding an ace increases the value of the cards that remain, since, for example, an opponent drawing to pocket aces has only half the usual chance of improving to 3 aces when the board is dealt [154]. ■

We have seen that the chance of pocket aces in hold'em is $\frac{1}{221}$. In pineapple, this probability rises, to

$$\frac{\binom{4}{2} \cdot 48 + 4}{\binom{52}{3}} = \frac{292}{22,100} \approx \frac{1}{76}.$$

This presumes, of course, that a player dealt a pair of aces and a nonmatching card holds the aces.

Once the third hole card in a hand of pineapple is discarded, the game is effectively hold'em, so a distinct pineapple strategy is centered around discarding. This requires careful attention to how many players stay in to see the flop. If a player is competing against many opponents, it may be a better move to break up a higher 2-card hand in order to pursue a hand that has a chance of improving to something far better. In Example 4.21, if the $A\spadesuit$ were replaced by the $A\diamondsuit$, it may be wise to hold $A\diamondsuit$ $K\diamondsuit$ in the hope of flopping a flush draw or straight draw. This is also possible when holding $A\diamondsuit$ $K\diamondsuit$ [141].

Straight draws are sometimes more complicated if discarding a hole card to pursue a straight draw means throwing away a card that might be needed to complete an eventual straight. For example, when dealt $Q\spadesuit$ $J\diamondsuit$ $9\heartsuit$, discarding the 9 gives a straight draw, but one of the 9s necessary to complete a $KQJT9$ or $QJT98$ straight is out of the deck and unavailable. This can also happen, more obviously, with flush draws when a player's 3 hole cards are all of the same suit. The chance of 3 suited hole cards is

$$\frac{4 \cdot \binom{13}{3}}{\binom{52}{3}} = \frac{22}{425} \approx .0518,$$

approximately 5%.

If a player's hole cards are all of one suit, the chance of flopping a flush draw drops from

$$\frac{\binom{11}{2} \cdot 39}{\binom{49}{3}} \approx .1164$$

to

$$\frac{\binom{10}{2} \cdot 39}{\binom{49}{3}} \approx .0953.$$

One card decreases the probability by 18%, and that probability was only about 11½% to begin with. Three suited cards might be too much of a good thing in pineapple, and unless there are high cards with a possible straight draw, folding before the flop might be the best choice [141].

Example 4.22. Suppose that you hold four cards to a diamond flush in pineapple following the flop and did not discard a fifth diamond. Find the probability of completing the flush on the turn or river.

Your hand becomes a flush if 1 or 2 of the final two cards is a diamond. There are 46 unknown cards to consider. The Complement Rule tells us that

$P(\text{Flush}) = 1 - P(0 \text{ diamonds on the turn and river})$. With 9 diamonds still out, this becomes

$$P(\text{Flush}) = 1 - \frac{\binom{37}{2}}{\binom{46}{2}} \approx .3565.$$

■

Variations

A number of pineapple variants have found their way to card rooms and home games.

- In *Crazy Pineapple Poker*, players may see the flop before discarding a hole card. Crazy pineapple may be played as a high/low game where the plot is split between the high hand and the low hand; if the low hand is not 8-high or better, the entire pot is awarded to the high hand [121].
- *Dirty Pineapple Poker* allows players to see the flop and the turn, but not the river, before discarding.
- *Lazy Pineapple Poker* makes the discard a formality by allowing players to postpone their discard until all 5 community cards are dealt. Each player's final hand can use no more than 2 hole cards.
- *Irish Poker* deals 4 hole cards to each player. Two must be discarded after the second round of betting, following the flop.
- *Watermelon* also deals each player 4 hole cards. One must be discarded before the flop, and a second discard made after the flop [14].

4.5 Prop Bets

As noted on page 107, *proposition* or *prop* bets are side bets made, often between players, on details that are peripheral to the main game. These are not uncommon in live hold'em games. One example is a bet between 2 players on the color of the 3 flop cards. If a majority of the cards must be of the winning player's color, this bet has a winner on every hand. A more exciting version might let the bet ride until all 3 flop cards are the same color, and might offer a higher payoff if the flop is suited. If made between players, this bet is fair. Other side bets offering odds other than 1–1, under dealer control or offered by hustlers, may favor one side.

Example 4.23. Some prop bets are not susceptible to crisp mathematical analysis. In March 2023, poker player Bill Perkins challenged fellow player Shaun Deeb to a prop bet that Deeb could reduce his body fat percentage to 17% by May 30, 2024. Perkins offered Deeb 10–1 odds, putting up $1 million against $100,000 from Deeb, who weighed 306 pounds when the wager was announced [18].

Calculating Deeb's probability of winning and the expected value of this bet to either bettor involves far too many variables for efficient computation—nonetheless, both players agreed to the wager. In February 2024, with Deeb's body fat percentage measured at 22.0%, Perkins offered to settle the bet early for $800,000, an offer which Deeb accepted [142]. ■

In 2011, Nevada sports books were first allowed to offer prop bets on the final table of 9 players at the World Series of Poker's Main Event: the $10,000 buy-in no-limit Texas hold'em championship. A very simple proposition allowed gamblers, including those not playing at the final table, to bet on whether more red or black cards would be dealt among the 3 cards comprising the flop in the first hand at the final table. Both red and black were posted at −110, which is shorthand notation meaning that players could win $100 for every $110 they wagered, or receive the same 10–11 odds on bets of other amounts [150]. (By contrast, a bet offered at a positive amount, such as +110, offers the positive payoff indicated for a winning $100 bet.)

This is, of course, a 50/50 proposition; the casino's goal in setting the prices at −110 was to encourage approximately equal action on both colors. For any two paired $110 bets, one on red and one on black, the casino would take in $220 and pay out $210 to the winner, guaranteeing a $10 profit regardless of which color won. The expected value of either bet is

$$E = (100) \cdot \frac{1}{2} + (-110) \cdot \frac{1}{2} = -\$5,$$

so the sports book held a 4.55% advantage.

Insurance

When playing blackjack, *insurance* is a prop bet offered by the house when the dealer's upcard is an ace. The bet may be made for up to half of a player's main bet and pays 2–1 if the dealer turns over a 10-count card for a natural 21. This wager got its name from the idea that a player could make it to "insure" a good initial hand and so incur less of a loss if the dealer beat every player not holding a natural.

In hold'em, insurance is a prop bet that might be made between 2 players who are faced off head-to-head in an all-in pot with one or more cards remaining to be dealt, or between one of the players and an outside observer. Since one player is all-in, there can be no further betting on the main hand, and each player typically turns their cards face up. At this point, the player with the best potential to win the hand might offer or ask for insurance. Alternately, an outside agent, perhaps a spectator or eliminated player who seeks to capitalize on a player's desire not to lose when holding very good cards, may offer an insurance bet to either player [11].

Unless a blackjack player is counting cards and knows he has the edge, blackjack insurance is usually a bad wager. Similarly, hold'em insurance is generally ill-advised, since the player placing the bet is typically offered less favorable odds than her odds of winning. For example, a player who is a 3–2 favorite based on the cards on the table might be offered a 13–10 insurance bet, which pays off only if she loses the hand. The bet might then be thought of as insurance against a bad beat. A \$100 insurance bet at these odds has expected value

$$E = (130) \cdot \frac{2}{5} + (-100) \cdot \frac{3}{5} = -\$8,$$

giving an 8% edge to the player or agent offering the bet. Part of the challenge to the agent wishing to approach a player and sell insurance is accurately evaluating the target player's chance of winning in the short time interval available.

Insurance could also be offered to the player with the lower chance of winning. The player above who is the 3–2 underdog might be offered 5–7 odds on an insurance bet. A \$70 insurance bet on these terms would give the agent an edge of 2.86%.

As with every other bet offered by casinos—or by life—if the game is not fair, the best move is to make sure it's unfair in your favor. For hold'em insurance, this means that it's better to be the one offering insurance than the one taking it, for that person has the chance to set the insurance odds at a level that will favor him or her. If the odds offered are not to the player's liking, there is no obligation to accept them. Of course, if the person offering insurance has made a miscalculation—as in offering 8–5 odds to the 3–2 favorite above—the player holds a 4% edge and should accept the terms as offered.

Run It Twice

A prop bet that two players can arrange if they're both all-in on a hand after the flop is *running it twice*. This prop bet is an agreement to split the total pot, deal two sets of turn and river cards, and award half the pot to the winning hand after each set of turn and river cards is dealt.

Suppose that Alex and Brett are all-in for a total pot of $\$D$ and that Alex has probability p of winning. Brett's chance of winning is then $1 - p$. If they play out the hand once, Alex's expectation is then pD while Brett's is $(1-p)D$. At this stage of the game, we need not consider how much each player has contributed to the total pot, nor need we be concerned about contributions from players who have since folded.

If we assume that the two runs of the turn and river are independent (which is close enough to correct for this analysis), it follows that Alex's expectation is

$$E(A) = p^2 \cdot D + 2p(1-p) \cdot \frac{D}{2} = p^2D + pD - p^2D = pD$$

—the same expected value. Similarly, Brett's expected value has not changed; it remains

$$E(B) = (1-p)^2 \cdot D + 2p(1-p) \cdot \frac{D}{2} = D - 2pD + p^2D + pD - p^2D = (1-p)D.$$

What has changed is each player's variance, which has decreased. Recall that the simplified formula for variance (page 44) is

$$\sigma^2 = E(X^2) - \mu^2 = \sum_x x^2 \cdot P(X = x) - \mu^2.$$

Both players have a variance of $D^2p(1-p)$ when playing the hand out once. Alex's variance when running it twice is

$$\sigma^2(D) = \frac{D^2}{4}(2p)(1-p) + D^2p^2 - p^2D^2 = \frac{D^2}{2}p(1-p) < D^2p(1-p),$$

while Brett's variance is

$$\sigma^2(B) = \frac{D^2}{4}(2p)(1-p) + D^2(1-p)^2 - (1-p)^2D^2 = \frac{D^2}{2}p(1-p) < D^2p(1-p),$$

so the variances for both players are the same as each other and less than the variance from just playing the hand once.

Variance measures the spread of results about the mean; the decrease in variance reflects the considerable potential for each player to win one run of the hand and collect $\frac{D}{2}$, an intermediate result between winning it all and losing it all. For a player with $p = \frac{1}{3}$, for example, the probability of losing both hands when running it twice is $\frac{4}{9}$—less than 50%, and considerably smaller than the $\frac{2}{3}$ chance of losing a single hand. Conversely, the player with $p = \frac{3}{4}$ can reduce his chance of losing everything from $\frac{1}{4}$ to $\frac{1}{16}$ by accepting a $\frac{3}{8}$ probability of winning only one of the two runs of the hand.

The Hot Poker Spot

Formal prop bets were a fixture on blackjack tables for years before the first casino-backed prop bet in an American poker room was introduced at Casino Del Sol in Tucson, Arizona in 2015. *The Hot Poker Spot* bet was installed at the casino's Texas hold'em tables, and paid out depending on the rank of the best 5-card poker hand made up of the player's two hole cards and three of the five community cards. This bet was between each player and the casino, and was not related to how the hand played out among the various players at the table. Both of the player's cards must be used to make the hand in order to win the prop bet, which wins if the hand is a straight or higher. Additionally, in order to win this bet, the player has to stay in the hand, not folding, until the final showdown.

The pay table for The Hot Poker Spot is shown in Table 4.6.

TABLE 4.6: The Hot Poker Spot pay table [2].

Hand	**Payoff**
Royal flush	1000 for 1
Straight flush	200 for 1
Four of a kind	100 for 1
Full house	25 for 1
Flush	15 for 1
Straight	10 for 1

Example 4.24. Under this rule, what is the probability of a qualifying royal flush?

The player's hole cards must both be part of a royal flush, and then the community cards must contain the other three cards, plus any two others. The order matters here: The probability that the first two cards are part of a royal flush is

$$p = 4 \cdot \frac{\binom{5}{2}}{\binom{52}{2}} = 4 \cdot \frac{5}{52} \cdot \frac{4}{51} = \frac{20}{663},$$

where the factor of 4 counts the possible suits. Once this has been achieved, the suit is determined and we need to draw the remaining three royal flush cards in the five community cards—it doesn't matter whether they're part of the flop, turn, or river. The probability of this occurring is

$$q = \frac{\binom{3}{3} \cdot \binom{47}{2}}{\binom{50}{5}} = \frac{1}{1960},$$

giving a total probability of

$$P(\text{Royal flush}) = pq = \frac{1}{64,974},$$

which is 10 times as great as the chance of drawing a royal flush in 5 cards. ■

The probability of a royal flush in 7 cards, without the restriction imposed by this wager, is

$$\frac{4 \cdot \binom{47}{2}}{\binom{52}{7}} = \frac{4324}{133,784,560} = \frac{1}{30,940},$$

so less than half of all 7-card royal flushes (which are rare to begin with) qualify for this bonus.

Player reaction to the Hot Poker Spot was mixed. Many veteran poker players objected; they viewed the prop bet as a way for casinos to extract more money directly from players' bankrolls without passing it through the poker table and giving other poker players a shot at winning it. Casino operators reported increased traffic in the poker room, ascribed to new poker players who were drawn to the new betting option.

Bet The Board

The hallmark of a good prop bet in any casino game is simplicity: the bet should be easy for players to understand and easy for dealers to administer without interfering with the play of the main game. By that benchmark, a prop bet proposed in 2014 easily qualifies. *Bet the Board* is a fitting name for this bet: a prop bet based on the poker hand created by the 5 community cards.

Bet the Board was described as a wager that might maintain interest in the game among players who had folded their main hands. It also had potential as a bet that could be easily adapted to online poker and, in the case of a live game, made available to spectators [126]. A Bet the Board wager pays off if the community cards form a hand ranking two pairs or higher according to Table 4.7.

TABLE 4.7: Bet the Board pay table [126].

Hand	**Payoff**
Royal flush	5000–1
Straight flush	2000–1
4 of a kind	250–1
Full house	75–1
Flush	50–1
Straight	25–1
3 of a kind	10–1
2 pairs	5–1

Consulting the hand frequencies in Table 1.2 (page 22) allows us to compute the probability of a Bet the Board wager winning:

$$\frac{4+36+624+3744+5108+10,200+54,912+123,552}{2,598,960} \approx .0763.$$

Combining the payoffs with the information in Table 1.2 shows that the house advantage for Bet the Board is 7.49%. The pay table can be adjusted to produce any desired house edge. We can do this by thinking of the expected value of a \$1 Bet the Board wager as a function of the payoff amounts $x_1 - x_{10}$ for each of the 10 recognized hands. Here, x_1 is the payoff for a royal flush, hand #1, x_2 is the straight flush payoff, and so on down to one pair and high card hands: $x_9 = x_{10} = -1$, which reflects the rule that the board must show at least 2 pairs for Bet the Board to win.

Label the hands in order as $h_1 - h_{10}$. We then have

$$E(x_1, x_2, \ldots, x_{10}) = \sum_{k=1}^{10} x_k \cdot P(h_k).$$

The Excel spreadsheet shown in Figure 4.6 was used to explore the effect on E of changing the payoffs $x_1 - x_8$, which are listed in the Payoff column.

In Figure 4.6, the expected value E is calculated using the **sumproduct()** function in Excel to multiply each hand's probability by its corresponding payoff and then to add the products together. Changing a payoff leads to a change in E; for example, reducing the payoff for 2 pairs to 4–1 raises the edge to 12.25%.

If the casino advertises a promotion where the Bet the Board payoff on a royal flush is raised to 25,000–1, the HA remains a healthy 4.41%, although there could be a short-term hit to the table's bankroll if a royal flush does show on the board.

Hand	Count	Probability	Payoff
Royal flush	4	1.539E-06	5000
Straight flush	36	1.385E-05	2000
4 of a kind	624	0.0002401	250
Full house	3744	0.0014406	75
Flush	5108	0.0019654	50
Straight	10200	0.0039246	25
3 of a kind	54912	0.0211285	10
2 pairs	123552	0.047539	5
Pair	1098240	0.422569	**-1**
High card	1302540	0.5011774	**-1**
		E	-0.0749
		HA	7.49%

FIGURE 4.6: Spreadsheet calculation of the Bet the Board expectation and house advantage.

Another Bet the Board possibility is to offer a progressive jackpot for the highest hands, perhaps for 4 of a kind or higher. A *progressive* jackpot is a prize that starts out at a fixed amount and grows by a small percentage of each bet until it is won by a player holding a specified rare hand. These are found on some video poker machines as well as in table games. If the jackpot gets sufficiently high, the edge shifts from the casino to the gamblers. Denoting the progressive jackpot by J, we find that the expected value of a \$1 Bet the Board wager is

$$E = (J) \cdot \frac{664}{2,598,960} - .1703,$$

where $-.1703$ represents the sum of the products of the probabilities of hands ranked lower than 4 of a kind with their respective payoffs, including a –1 payoff if the board shows a pair or a high-card hand. Solving the equation $E = 0$ for J shows that the expectation is positive for a single bettor if $J \geqslant \$666.72$.

If the progressive jackpot must be divided among multiple winning gamblers, it is necessary to multiply J by the number of players who share in the prize.

Zotak Poker

Zotak Poker is a constellation of 11 prop bets, originally launched in the United Kingdom in 2015, that are resolved by the 2 burn cards discarded before the turn and the river [107]. After the hand is complete, the burn cards are retrieved and turned up, and the bets are settled. Players may bet on a variety of options:

- Several bets pay off if the two burn cards form a specified pair.
- Additional betting options center on the color of the burn cards.
- A third class of bets may be made on the blackjack value of the burn cards, with aces always counting as 1, face cards as 10, and other cards their numerical value.

Zotak Poker bets and their payoffs are listed in Table 4.8. For any given pair of burn cards, at least 1 bet must win, as one of the “total” bets wins on every hand unless the burn cards are a pair of aces, in which case the “pair of aces or kings” bet is a winner. It is possible for as many as 4 Zotak Poker bets to win on the same pair of burn cards, for example, if the cards come up 9♡ 9♢.

Zotak Poker bets push if the hand ends before the flop. If the hand ends before the turn or river are dealt, the cards necessary to resolve the prop bets are dealt and all bets have action.

TABLE 4.8: Zotak Poker wagering options [107].

Wager	**Payoff**
Pair of 2s	200–1
Pair of aces or kings	100–1
Pair of 9s through queens	50–1
Both burn cards red	3–1
Both burn cards black	3–1
Both burn cards the same color	1–1
Total of 8 or 9	10–1
Total of 4, 5, 6, 8, or 10	5–1
Total of 3, 7, 9, or 16	5–1
Total of 15, 16, 18, 19, or 20	2–1
Total of 11–15 or 17	1–1

The probability of a pair of a given rank is $\frac{1}{221}$, so the expectation of the pair of 2s bet is

$$E = (200) \cdot \frac{1}{221} + (-1) \cdot \frac{220}{221} = -\frac{20}{221} \approx -\$.0905,$$

and its HA is 9.05%. Other pair bets have very similar expectations due to their symmetry: the number of winning combinations doubles as the payoff is halved. The bet on a pair of aces or kings has a HA of 8.60% and the bet on a pair of 9s through queens carries a house edge of 7.69%.

A better deal for gamblers is the trio of color bets. The probability of winning the bet that both cards are the same color is $p = \frac{25}{51}$: the first card can be either color, and then the second card, chosen without replacement, must match its color. For bets paying even money with win probability p, the expected value is

$$E = (1) \cdot p + (-1) \cdot (1 - p) = 2p - 1,$$

giving an expectation of –.0196 and a house edge of only 1.96%.

Assessing the bets on the sum of the burn cards is a simple matter of considering the number of ways to make that sum in 2 cards. If the sum is reachable through a pair, the probability is $\frac{1}{221}$ unless that sum is 20, in which case the probability is $\frac{20}{221}$. The probability of a sum that is made through two cards of different ranks is $\frac{1}{169}$ if neither card counts 10; a 10-count card with a non-10 has probability $\frac{4}{169}$.

Example 4.25. A total of 12 can be formed in the following ways:

$$2 + 10,\ 3 + 9,\ 4 + 8,\ 5 + 7,\ 6 + 6.$$

All but 6+6 can be completed with the 2 cards in either order. The probability of 2 burn cards adding up to 12 is then

$$2 \cdot \frac{4}{169} + 2 \cdot \frac{1}{169} + 2 \cdot \frac{1}{169} + 2 \cdot \frac{1}{169} + \frac{1}{221} = \frac{251}{2873} \approx .0874.$$

■

4.6 Exercises

Answers to starred exercises begin on page 337.

Texas Hold'em

4.1. In the episode "New Year's Eve" from the fourth season of the TV show *Modern Family*, a game of Texas hold'em culminates in the following three-player showdown:

- Eleanor: Aces and 8s (two pairs).
- Billy Dee: 3 jacks.
- Jay: Straight to the queen ($89TJQ$).

Reconstruct the player hands and a 5-card board that could lead to this result.

4.2. Is it possible, at a 9-player Texas hold'em table, for each player to hold one of the 9 hand types ranked 1 pair or higher simultaneously? Either exhibit a board and 9 player hands that achieve this or explain why it is impossible.

4.3.* Suppose that you are dealt pocket kings. Find the probability that the opponent to your immediate right has pocket aces.

4.4.* David Sklansky's list of 72 viable hole-card holdings separates the hands into 8 groups [140]. The top group includes AA, KK, QQ, and AK suited or unsuited. Find the probability of receiving one of these combinations as your hole cards.

4.5.* A *perfect catch* is a hold'em hand where there are only 2 cards that can turn a player's post-flop hand into the winning hand on the turn and the river—and both cards appear [11]. An example of a hand where a perfect catch is possible is when Player A holds $A\diamondsuit$ $A\clubsuit$, Player B holds $K\spadesuit$ $Q\spadesuit$, and the flop comes $A\spadesuit$ $A\heartsuit$ $6\spadesuit$. Player A holds 4 aces; Player B can only win if the turn and the river are the jack and 10 of spades, in either order.

Find the probability that a player in a perfect catch situation against one opponent gets both of the cards that he or she needs to win.

4.6.* If your hole cards are $K\diamondsuit$ $7\spadesuit$ and the flop is $8\spadesuit$ $6\spadesuit$ $5\spadesuit$, how many outs will turn your hand into a straight, a flush, or a straight flush? Find the probability of success and its Rule of Four approximation.

4.7.* In 1991, Goldberg's Casino in Deadwood, South Dakota offered the *Montana Banana Bonus* at its 7-card stud and hold'em tables [28]. Any player holding either 2 pairs, deuces and 9s, or a full house consisting of deuces and 9s won a $13 bonus. Note that this bonus was a promotion that required no player wager, and so its expectation was positive. Find the probability of a 5-card hand that wins the Montana Banana Bonus.

4.8.* On the days when the Orleans was not running the First Five bonus, the Suited Royals bonus was active. This promotion paid $250 to a player making a suited royal flush during Texas hold'em play. Separate jackpots were awarded for royals in each of the 4 suits; if one suit jackpot was not won on a given promotion day, $250 was added to the payoff for the next day. Winning hands were required to use both of a player's hole cards to form the royal flush.

a. Consider a 7-card hold'em hand that contains a royal flush. What is the probability that the hole cards are both part of the royal flush?

b. Find the probability of a winning Suited Royals hand.

c. What is the minimum payoff (which must be a multiple of $250) for this bonus to have an expected value exceeding 10¢?

4.9.* Suppose that your hole cards in a hand of hold'em are a pair of aces and that the flop is an ace together with 2 unmatched cards. You hold 3 aces, and might only fear being beaten by an opponent drawing into a straight. What is the probability that the turn produces a 4-card board where someone can hold a straight? (Hint: Remember that any 5-card straight dealt from a standard deck must contain either a 5 or a 10.)

4.10.* One way to flop into a double-belly-buster straight is to be dealt hole cards that are exactly 6 apart in rank, such as $J5$. Find the probability of hole cards separated by 6.

4.11.* An *overpair* is a pair of hole cards that ranks higher than any of the 3 community cards. The potential for an overpair gives some additional value to certain pairs below QQ. If your hole cards are TT, what is the probability that the flop consists of 3 cards of different ranks that give you an overpair?

Short Deck

4.12.*

a. If your hole cards are $7\heartsuit$ $Q\heartsuit$, find the probability that the flop completes a flush in both standard Texas Hold 'Em and in Short Deck.

b. Find the probability of flopping a flush draw (page 140) in Short Deck.

4.13.* How many high-card hands, considering all 7 cards, are possible in Short Deck?

4.14.* When the 5 community cards include 2 or more cards of the same rank, this is called *pairing the board*. Find the probability that the board in Short Deck contains exactly one pair (so not 2 pairs or higher).

4.15. Confirm that the Rule of Three is a good approximation to the actual probability of completing a 6-card Short Deck hand in a 1-card draw with k outs.

4.16.* Find a value of N so that a Rule of N for Royal Hold'em gives a good approximation for the probability of improving a hand with k outs when drawing the turn and river, where $1 \leqslant k \leqslant 7$.

Omaha

4.17. In most popular forms of poker using 5-card hands, including Texas hold'em, it's possible to have a tie between 2 royal flushes. Explain why this cannot happen in Omaha.

4.18.* Find the probability that your hole cards in a hand of Omaha contain 1 card from each suit.

4.19.* An attractive set of hole cards in Omaha high-low is "double-suited aces": a pair of aces and two other cards of the same suits as the aces. Find the probability of being dealt double-suited aces.

4.20.* Find the probability that hole cards of $A\spadesuit\ 2\spadesuit\ J\diamondsuit\ Q\heartsuit$ are counterfeited when the full 5-card board is revealed.

4.21. The Hot Poker Spot prop bet is also available for Omaha tables. Payoffs are lower for the Omaha bet because players have 4 hole cards to choose 2 from in making a high hand. The pay table for this bet is shown in Table 4.9.

TABLE 4.9: The Hot Poker Spot pay table for use at an Omaha table [76].

Hand	**Payoff**
Royal flush	500 for 1
Straight flush	30 for 1
Four of a kind	20 for 1
Full house	8 for 1
Flush	5 for 1

Find the probability of a four-of-a-kind hand in Omaha that qualifies for the 20 for 1 payoff.

Pineapple Poker

4.22.* In [121], a discussion of crazy pineapple high/low starting hands that have sufficient potential to be played through the flop ends with AKx with the ace and x suited and the king of a different suit. If x is a 9 or higher, the hand can improve to a good high hand on the flop; if $x \leqslant 8$, then the hand has 2 suited cards to a possible low hand. In how many ways may this 3-card hand be dealt?

4.23. Find the probability that a pineapple poker player holds a pocket pair and flops a set (Example 4.17).

4.24.* Suppose that your hole cards in a hand of pineapple poker are $A\diamondsuit$ $K\diamondsuit$ 3$\clubsuit$ and the flop comes $A\spadesuit$ $K\heartsuit$ 5$\clubsuit$, so you discard the 3$\clubsuit$ and play on with the top 2-pair hand. Find the Rule of Four approximation to your chance of filling out to a full house and compare it to the actual probability.

Prop Bets

4.25.* Suppose that two players, both all-in after the flop, agree to run it *three* times instead of twice. Each player has the same variance for this game. Calculate it.

4.26.* Find the expected value of the Zotak Poker bet that the sum of the burn cards is 8 or 9.

4.27.* In Zotak Poker, the two bets that the burn cards are red and black have the same house advantage. Find it.

Chapter 5

Advanced Card Counting

Prior to considering variations on stud, draw, and hold'em poker games, including both games where players face off against each other and casino table games with a poker flavor where players match their hands against a dealer's hand or a fixed pay table, we shall pause briefly to consider some additional questions that arise in counting poker hands. These questions arise when we consider poker hands beyond the ones we've looked at so far, variations on the standard 52-card deck, or changes in the size of a hand.

5.1 Why Five?

One might reasonably ask why most live poker games have settled on 5 cards as the eventual size of a hand dealt from a standard 52-card deck. Hands with 3 or 4 cards might be simpler to analyze, and have been used as the basis for several poker-based casino carnival games. Increasing the number of cards to 6 or 7 includes the possibility of some additional hands, such as 3 pairs, or 4 of a kind plus 3 of a kind. Of course, dealing larger hands, particularly in a draw game, decreases the number of players who can participate at one time unless a deck with more than 52 cards is used.

Some principles that might be appropriate in assessing the "best" hand size are these:

1. **"High card" should be the lowest-ranked hand.** A hand size where "high card" is less common than a pair is counterintuitive. This becomes an issue when the number of cards in a hand increases, since pairs become more likely and it gets harder to draw a hand without one.

   It is, of course, possible to force high-card hands to be lowest-ranked by disregarding probabilities when ranking hands and simply declaring that hands without a pair, a straight, or a flush rank at the bottom.

2. **There should be no unbeatable hand.** As noted on page 3, an unbeatable hand is thought by some to go against the spirit of gambling. This criterion favors the straight flush or royal flush as the best possible hand, since those hands can, depending on the game being played, often

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be tied by another hand of the same cards in a different suit. In a 5-card game, four aces, or four kings and an ace, is unbeatable without the straight flush as a recognized hand.

3. **Within the constraints of the first two principles, there should be a robust selection of different hands.** In general, this suggests that larger hands are preferable. In a poker game played with 2-card hands, the only possible hands would be straight flushes, pairs, flushes, straights, and high-card hands, which might not provide enough variety to be interesting. That said, Hurricane (page 195) is a recognized poker game using 2-card hands.

To address this question, we return to the master formula introduced on page 22 for counting hands in a 52-card deck [22]. The number of ways to draw exactly h sets of i cards of the same rank and j sets of k cards of the same rank among a hand of m cards, where $hi + jk \leqslant m$, is

$$\binom{13}{h}\binom{4}{i}^h \cdot \binom{13-h}{j}\binom{4}{k}^j \cdot \binom{13-h-j}{m-hi-jk}\binom{4}{1}^{m-hi-jk}.$$

Previously, we have focused on the case $m = 5$; we now generalize to hands containing different numbers of cards. To this formula we append the following general formulas that count straights, flushes, and straight flushes:

$$\begin{aligned}\text{Straight flushes:}\quad & 4(15-m)\\ \text{Flushes:}\quad & 4\binom{13}{m} - 4(15-m)\\ \text{Straights:}\quad & (15-m)\binom{4}{1}^m - 4(15-m).\end{aligned}$$

Table 5.1 shows the probability of various poker hands when 3–7 cards comprise a hand. In this table, royal flushes are included with straight flushes.

There are some noteworthy trends apparent in the table.

- The probability of a pair increases as the hand size increases until it reaches 7 cards, while the probability of a high-card hand decreases. This has implications for the first principle above.

- With regard to the second principle, straight flushes are properly the highest-ranked hand unless hands contain 4 cards. Four aces is unbeatable in 4-card poker. Two straight flushes may tie if they contain the same 5 consecutive ranks in different suits, so straight flushes are not unbeatable in 5-card poker.

- Hands which are impossible with 5 or fewer cards are nonetheless rare, so the third principle may not have much practical use unless the game is draw poker. However, draw poker decreases the maximum number of players. A 7-card game can reasonably accommodate no more than

TABLE 5.1: Hand probabilities: 3–7-card hands [22].

Hand	Cards in hand **3**	**4**	**5**	**6**	**7**
Pair	.1694	.3042	.4223	.4855	.4728
2 pairs	——	.0104	.0475	.1214	.2216
3 pairs	——	——	——	.0030	.0185
3 of a kind	.0024	.0092	.0211	.0360	.0493
Two 3 of a kinds	——	——	——	.0001	.0004
4 of a kind	——	.00005	.0002	.0007	.0014
Full house	——	——	.0014	.0081	.0246
Pair & 4 of a kind	——	——	——	.00004	.0003
2 pairs & 3 of a kind	——	——	——	——	.0009
3 of a kind & 4 of a kind	——	——	——	——	.000005
Straight flush	.0022	.0002	.00002	.000002	.0000002
Flush	.0496	.0104	.0020	.0003	.00005
Straight	.0326	.0102	.0040	.0018	.0010
High card	.7439	.6553	.5012	.3431	.2091

4 players with a standard deck unless discards are recycled, whereas 7-card hands without a draw allow as many as 7 players.

- The probability of flushes, straights, and straight flushes decreases as the hand size increases, reflecting the increasing difficulty of collecting larger sets of cards of the same suit or of consecutive ranks.

- The column for 4-card hands reveals an unusual feature that we first noted on page 94: 2-pair hands, straights, and flushes are almost equally likely. Using the formulas above, we see that there are 2772 straights, 2808 two-pair hands, and 2816 flushes when 4 cards are dealt to each hand. While the order "a straight beats 2 pairs, which beats a flush" is mathematically correct, the three hands are so close in frequency that there is scarcely any distinction.

Table 5.1 together with the three principles listed above make a strong argument that the optimal size for a poker hand is 5 cards.

5.2 Hand Rankings Revisited

Designation of recognized poker hands is somewhat arbitrary, as we note in the fact that straights and straight flushes were not fully accepted until the early 20th century. In a game without straights or straight flushes, the relative hand rankings in Table 1.2 do not change. The 40 straight and royal flushes are added into the flush total, giving 5148 flushes—still fewer than the number of three of a kinds. The 10,200 straights become high-card hands, which raises their number without disrupting the rankings.

A closer look at hand frequencies reveals that the division of stud poker hands, where there is no opportunity to discard and replace cards, into these 10 standard categories masks some interesting differences in the relative scarcity of some hands within some categories. Among two-pair hands, for example, the rarest hand is "3s up", which must be a pair of 3s and a pair of 2s. The probability of 3s up is

$$\frac{\binom{4}{2}\cdot\binom{4}{2}\cdot 44}{\binom{52}{5}} = \frac{1584}{2,598,960},$$

and this is beaten by "aces up", even though the probability of aces up is 12 times higher:

$$\frac{\binom{4}{2}\cdot 12\cdot\binom{4}{2}\cdot 44}{\binom{52}{5}} = \frac{19,008}{2,598,960}.$$

In general, the number of Xs up hands, where X denotes any rank from 3 through ace, is

$$\binom{4}{2}\cdot(X-2)\cdot\binom{4}{2}\cdot 44,$$

where jacks through aces have $X = 11, 12, 13, 14$.

This condition where differently-rare hands are combined into a single hand type where rarer hands lose out to more common hands of the same type also extends to flushes and high-card hands.

A complete accounting of all 35 hands, in properly ranked order from highest to lowest and with flushes, two-pair hands, and high-card hands broken out by their individual scarcity, may be found in Table 5.2 on page 158. We see that 1-pair hands, which number 84,880 regardless of the rank of the pair and so may rightly be collected together as a single hand type, rank lowest using this ranking system.

Example 5.1. Using Table 5.2 to rank hands shows that the second-ranked hand, right behind the 4 royal flushes and just ahead of the 36 straight flushes, is a 7-high flush. Such a hand must be $76542, 76532, 76432$, or 75432. Each hand can occur in any of the 4 suits, giving a total of 16 7-high flushes.

The number of flushes with high card X, with X extended to face cards—but not aces—as above, is

$$4\left[\binom{X-2}{4}-1\right].$$

TABLE 5.2: 35 stud poker hand types, ranked by scarcity [36].

Hand	**Count**	**Probability**
Royal flush	4	1.539×10^{-6}
7-high flush	16	6.156×10^{-6}
Straight flush	36	1.385×10^{-5}
8-high flush	56	2.155×10^{-5}
9-high flush	136	5.233×10^{-5}
10-high flush	276	1.062×10^{-4}
Jack-high flush	500	1.924×10^{-4}
4 of a kind	624	2.401×10^{-4}
Queen-high flush	836	3.217×10^{-4}
King-high flush	1316	5.064×10^{-4}
3s up	1584	6.095×10^{-4}
Ace-high flush	1972	7.588×10^{-4}
4s up	3168	.0012
Full house	3744	.0014
7-high	4080	.0016
5s up	4752	.0018
6s up	6336	.0024
7s up	7920	.0030
8s up	9504	.0037
Straight	10,200	.0039
9s up	11,088	.0043
10s up	12,672	.0049
Jacks up	14,256	.00549
8-high	14,280	.00550
Queens up	15,840	.0061
Kings up	17,424	.0067
Aces up	19,008	.0073
9-high	34,680	.0133
3 of a kind	54,912	.0211
10-high	70,380	.0271
Jack-high	127,500	.0491
Queen-high	213,180	.0820
King-high	335,580	.1291
Ace-high	502,860	.1935
1 pair	1,098,240	.4226

The factor of 4 chooses the suit of the X, and the factor in brackets counts the number of ways to choose the other 4 ranks, with 1 subtracted to remove straight flushes. If $X = 14$, corresponding to an ace-high flush, this formula must be adjusted, to

$$4\left[\binom{X-2}{4} - 2\right],$$

since both $AKQJT$ and $5432A$ must be excluded. ■

High-card hands are counted similarly, by accounting for the ranks and suits separately and then multiplying the two factors. For $7 \leqslant X \leqslant 13$, there are

$$\left[\binom{X-2}{4} - 1\right] \cdot \left(4^5 - 4\right)$$

X-high hands. Again, ace-high hands require a different related formula:

$$\left[\binom{X-2}{4} - 2\right] \cdot \left(4^5 - 4\right).$$

5.3 Extra Possibilities

Some local versions of poker have recognized hands beyond the list of 10 that we've been using, and some hand types that were occasionally identified in the past have faded from play. In 1895, an article from the New York *Herald* was reprinted across the country echoing the call from several years earlier (page 4) for a Poker Congress to standardize the game's rules. Among its arguments was a passage excoriating parts of the American West for barring straights, as well as a paragraph drawing a sharp line against further expansion of recognized poker hands, whose illegitimacy was declared in no uncertain terms.

> *On the other hand, skips and "blazes" and "tigers" and "Hibernian straights" and a score of other uncouth and imbecile combinations, whose very names evoke a shudder in every true lover of the game, have found their advocates in various parts of the country.*
>
> —New York *Herald*, 1895 [70].

We shall consider some of these uncouth combinations in this section. One such hand is a *blaze*, which is simply any hand consisting of 5 face cards [43]. The number of blazes is

$$\binom{12}{5} = 792,$$

formally ranking the blaze between a full house and four of a kind.

However, treating blazes this way overlooks the fact that a hand could have 4 of a kind and still be a blaze, such as $QQQQK$, so counting it as a blaze would be a disadvantage. At the same time, the minimum rank of a blaze is two pairs, like $QQJJK$. Accordingly, common practice in games using blazes is to rule that a blaze beats two pairs but loses to 3 of a kind, and that a blaze containing 3 of a kind or higher need not be called a blaze [43]. The advantage comes when a hand such as $QQJJK$ beats a higher-ranked two-pair hand like $AA559$.

With this understanding, the number of blazes that are actually called a blaze is precisely the number of 2-pair blazes:

$$\binom{3}{2} \cdot \binom{4}{2}^2 \cdot 4 = 432.$$

This is 54.5% of all blazes. The complete distribution of blazes is shown in Table 5.3.

TABLE 5.3: Distribution of blazes.

Hand	**Number of blazes**
Four of a kind	24
Full house	144
Three of a kind	192
Two pairs	432

Note that valuing blazes strictly according to their scarcity would mean that a blaze should beat a full house but lose to 4 of a kind, with an exception made so that the 24 blazes containing 4 of a kind should be counted as the higher hand.

Example 5.2. While the blaze takes hands of at least two pairs and revalues them, another alternate hand called the *skeet* takes some high-card hands and elevates them in the table of hand rankings [43]. A skeet consists of a 9, a 5, a 2, one card between the 9 and the 5, and one card ranking between the 5 and 2. If all 5 cards are of the same suit, the hand is called a *skeet flush*; see Figure 5.1.

FIGURE 5.1: A skeet flush, in spades.

Skeet flushes are easily counted using the Fundamental Counting Principle and accounting for suits: there are $4 \cdot (1 \cdot 3 \cdot 1 \cdot 2 \cdot 1) = 24$ of them. Accordingly, a skeet flush outranks a straight flush when it's accepted as a valid type of hand.

For skeets, there are 4 choices for each of the 3 fixed cards. The card between the 9 and 5 must be a 6, 7, or 8, and may be chosen in 12 ways. The card between the 5 and 2 is either a 3 or 4, so there are 8 choices. The total number of skeets is

$$4 \cdot (12 \cdot 4 \cdot 8 \cdot 4) - 24 = 6120,$$

where the 24 skeet flushes are subtracted at the end. This frequency means that a skeet beats a straight but loses to a flush. The removal of 6120 high card hands, for a skeet flush is still a flush and not a high card hand, does not change the ranking at the bottom of the list of hands; a pair still beats a high card hand. ■

It should be noted that, with 2,598,960 hands possible, there are many other nonstandard hands that could be identified, counted, and ranked among the standard 10 hand types. However, a nonstandard hand will only get attention if it's somehow interesting to players. While it's true that there are 6144 *skip straights* or *skips*: straights where the cards are in sequence and separated in rank by 2 each, such as $2468T$ or $579JK$, this hand has not caught on with most poker players. This number includes 24 skip straight flushes, which also garner little or no interest—though one authority claimed that a skip straight flush would beat a royal flush despite being 6 times more probable [123].

The line between standard and nonstandard hands was still in flux in the late 19th century. Straights were subject then to much of the same scrutiny as some hands not commonly recognized today. John W. Keller, who strongly advocated for straights as a legitimate poker hand in 1887, was less charitable toward other emerging hands, declaring that skip straights "add nothing to the game" and blazes were "the most contemptible of all poker innovations" [87]. John Blackbridge had earlier expressed the same thought in 1880, stating that the blaze is "destitute of any claims to respect" and that skip straights "are equally destitute of merit". Blackbridge went on to assert that the game of draw poker was complete as it stood in 1880 and that further attempts to modify the rules or add new hands would be worthless, much as chess variations had been abandoned and chess was accepted in 1880 as a finished game [8].

Nonetheless, a 1905 book by F.R. Ritter listed the following additional hands, describing them as a collection of modern additions to draw poker [123].

- *49*: A hand containing a 4, a 9, and three cards in between them without a pair.
- *Little Dog*: A hand containing a 2, a 7, and three cards in between without a pair or flush. Some authors refer to this hand as a *tiger*.

- *Big Dog*: A 9, an ace, and 3 cards between them, again without a pair or flush.
- *Little Cat*: Also known as an "83", this has a 3, an 8, and three cards between them without a pair.
- *Big Cat*: An 8, a king, and 3 cards between them without a pair.
- If any of these hands does constitute a flush, it is promoted to "[Name] Flush" and ranks much higher.

These extra hands were said to rank in the decreasing order Big Cat, Little Cat, Big Dog, Little Dog, 49, and Skip Straight. The corresponding flushes follow the same order.

Example 5.3. The roster of extra hands that we have examined here includes only hands that would be high card hands otherwise, except for the blaze. In a (complicated) game where all 7 extra hands are recognized, is "high card" still the lowest-ranked hand?

Up front, we note that no 5-card hand can fall into more than one of these types, so we can answer this question by simply counting each type and adding the results; there is no need to subtract any hands to avoid double-counting.

There are 6120 skeets and 6144 skip straights; the method used to count them is equally applicable to 49s and the various dogs and cats. We do not count flushes here, since a 49 flush—for example—would still be counted as a flush if 49s were not recognized.

For each of the 5 remaining hands, the ranks of the high and low cards are fixed. The ranks of the cards in between may be chosen in $\binom{4}{3} = 4$ ways, which gives 4 ways in total to select the 5 ranks. Suits may then be filled in $4^5 - 4$ ways: 4 choices for the suit of each card, minus 4 to remove the flushes.

Adding up over all 5 hands gives

$$5 \cdot 4 \cdot \left(4^5 - 4\right) = 20,400$$

hands that are either a 49 or some type of dog or cat. Adding in the skeets and skip straights gives 32,664 of these nonstandard hands, leaving 1,269,876 high card hands. This is still greater than the number of 1-pair hands: 1,098,240, so there is no need to adjust the rankings for standard hands when incorporating any—or all—of these options. ■

5.4 Historical Games

Any attempt to write a history of poker will invariably find that much of the game's past is shrouded in mystery. Various sources trace the game's origins to the British Isles, to continental Europe, and to Persia. Knowing a comprehensive history of poker is not necessary to appreciate the role of mathematics is the numerous variations of the game—which is fortunate, since no such definitive history exists.

A feature of card games collected under the "poker" name over the years is that players build a hand of playing cards from a larger deck, and then compete to see whose hand is the best, however the rules of the game define "best". The composition of the deck, which affects the relative scarcity of different recognized hands, has changed frequently over time. While the modern 52-card deck with 13 cards in each of 4 suits is now the standard, other decks have been used in the past and some games today use different decks.

Three-Card Brag

One of the earliest games with elements that are present in modern poker is *Three-Card Brag*, which dates back to at least the 16th century in Great Britain. Three-card brag is played with a 52-card deck; as the name suggests, players are dealt 3-card hands and bet against each other, with the highest hand after all remaining players have contributed equally to the pot winning. The game provided no opportunity to exchange cards as in draw poker; each player's original dealt hand was also their finishing hand.

In three-card brag, there are $\binom{52}{3} = 22,100$ hands in total. The recognized hands are ranked in the following order:

- The highest hand is 3 of a kind, three cards of the same rank, which is called a *prial*: short for "pair royal". Three 3s is the highest hand, outranking even 3 aces. Both 333 and AAA are equally likely, so this ranking respects mathematical frequency.

  The probability of a dealt prial is

  $$p = \frac{13 \cdot \binom{4}{3}}{\binom{52}{3}} = \frac{52}{22,100}.$$

  Dividing by 13 gives the probability of any specific prial, which is $\frac{4}{22,100} = \frac{1}{5525}$.

- 3-card straight flushes, 3 cards of the same suit in numerical sequence, rank next. These were called *running flushes*. Once again, the 3s are elevated: the highest-ranked running flush is a suited $32A$.

- Straights or *runs*, 3 cards in sequence but of different suits, were next. $32A$ is the highest-ranking run.

- In three-card brag, runs outrank *flushes*, which are next in line and consist of any 3 cards of the same suit but not in sequence.
- *Pairs*: two cards of the same rank and a third, unmatched card, rank below flushes. However, a pair of aces, not 3s, is the highest-ranking pair.
- *High card*—a hand with no prials, pairs, runs, or flushes—was the lowest-ranked hand.

When counting the various 3-card hands, an immediate discrepancy surfaces. As we saw above, the number of prials is

$$13 \cdot \binom{4}{3} = 52,$$

a result which can also be derived by thinking that each prial involves 3 of the 4 cards of one rank, so choosing the cards that form a prial is the same as choosing which card of the selected rank to omit. Since there are 52 cards in the deck, there are the same number of prials.

Running flushes can be counted as we did for 5-card straight flushes, by starting from the lowest-ranked card in the hand. Any card except a king can be the lowest card in a running flush, which makes 48 running flushes. This is smaller than the number of prials, even though prials outrank running flushes. This order would be corrected later in 3-Card Poker (page 252), where a straight flush properly outranks 3 of a kind.

Brag According to Hoyle

Englishman Edmond Hoyle (1672–1769) stands as an authority almost beyond reproach in card games. Although Hoyle died before poker was established as a game in its own right, the phrase "according to Hoyle" is sometimes cited as justification for one set of rules or another. Books bearing Hoyle's name continue to be published to the present day.

Hoyle wrote *A Short Treatise of the Game of Brag* in 1751 [78]. His version of brag is rather different from the game of three-card brag considered above. Most notably, Hoyle's version of brag incorporates 3 wild cards, called *braggers*: the $A\diamondsuit$, $J\clubsuit$, and $9\diamondsuit$. Braggers are ranked in the order listed, which aids in breaking ties among hands. Natural hands outranked corresponding hands with braggers, so the highest-ranked hand was 3 aces, followed by 2 aces and a bragger, 2 braggers and an ace, 3 braggers, and so on [78]. Only pairs and three-of-a-kind hands were recognized in this version of brag—flushes and straights had no standing as special hands.

The probability of a 3-card hand with at least 1 bragger may be found using the Complement Rule:

$$P(\geqslant 1 \text{ bragger}) = 1 - P(\text{No braggers}) = 1 - \frac{\binom{49}{3}}{\binom{52}{3}} \approx .1663,$$

very close to 1 chance in 6.

American Brag, by contrast, is played with 8 braggers: all of the jacks and 9s are wild [33]. The 8 braggers are not ordered; except for the fact that three jacks beat three 9s, all braggers are equal. Also different from brag is the American brag rule that bragger hands outrank natural hands, so JJJ is an unbeatable hand.

With 8 braggers instead of 3, the probability of being dealt 1 or more increases from 1 in 6 to

$$1 - \frac{\binom{44}{3}}{\binom{52}{3}} \approx .4007;$$

about 2 in 5.

As Nas and Straight Poker

As Nas is a card game of Persian origin with some similarities to poker. The game uses a 20- or 25-card deck consisting of 4 or 5 copies of 5 ranks of cards [137]. Suits are not used. In descending order, the cards are

- *As,* or ace.
- *Shah,* or king.
- *Bibi,* or lady, equivalent to the modern queen.
- *Serbaz,* or soldier.
- *Couli,* or dancer.

When dealt from a 20-card deck, as nas is similar to royal hold'em; the big difference being the absence of suits. As nas hands contain 5 cards. The highest possible hand in a 20-card deck is 4 of a kind; in a 25-card deck, it's 5 of a kind. Since there are no suits, there can be no flushes, and straights are not recognized; a hand dealt one card of each rank is regarded as an ace-high hand. As nas was a 4-player game using all 20 cards; a fifth player could be accommodated by using the larger deck.

With its 20 or 25-card deck, as nas has different hand frequencies and a different rank order of hands. For convenience, assume that as nas is being played with k cards of each rank, where k is either 4 or 5. This allows us to compute hand frequencies for both decks with a single formula, a function of k, for each hand. Table 5.4 shows the formulas and the counts for both decks.

The omission of straights in as nas has the effect of taking a relatively rare hand and moving it to the bottom of the rankings. In a 20-card deck, there are 1024 no-pair hands, and in a 25-card deck, 3125. In both decks, a straight or no-pair hand would properly rank between 3 of a kind and a full house.

The first game that bore the name "poker" is a variation on as nas using a 20-card deck: aces through 10s as we now recognize the cards. This game,

TABLE 5.4: As nas hand frequencies.

Hand	**Formula**	**20 cards** $\boldsymbol{k=4}$	**25 cards** $\boldsymbol{k=5}$
5 of a kind	$k\binom{k}{5}$	0	5
4 of a kind	$5\binom{k}{4}\cdot 4k$	80	500
Full house	$5\binom{k}{3}\cdot 4\binom{k}{2}$	480	2000
3 of a kind	$5\binom{k}{3}\cdot\binom{4}{2}\cdot 4^2$	1920	7500
2 pairs	$\binom{5}{2}\binom{k}{2}^2\cdot 3k$	7320	15,000
Pair	$5\binom{k}{2}\binom{4}{3}\cdot k^3$	7680	25,000
No pair	k^5	1024	3125

sometimes now called *straight poker*, dealt 5-card hands to each player [40]. There was no draw, nor any exposed cards, so betting became a matter of each player trying to convince the others that their hand was the strongest. Some observers have ventured a timeline where brag led to straight poker, which in turn led to draw poker [24].

Example 5.4. Straight poker got its start before the straight and straight flush were recognized as valid 5-card hand types. In [24], George Sturgis Coffin notes that holding an unbeatable hand of 4 aces or 4 kings with an ace seldom happens when the 20-card deck is fairly dealt [24]. Find the probability of an unbeatable hand.

There are 16 ways to draw 4 aces with a fifth card and 4 hands consisting of 4 kings and an ace, making the probability of an unbeatable hand

$$\frac{20}{\binom{20}{5}} = \frac{20}{15,504} \approx .0013.$$

This is approximately 1 hand in 774, which may fairly be described as "seldom". ■

Senatorial Poker: The Loo Loo

In the late 19th century, a poker variant casually called "the loo loo game" was said to be popular among members of the United States Congress. Loo loo used a 53-card deck, with one fully wild joker. The game added a number of nonstandard hands to the rankings, which are shown in order in Table 5.5.

TABLE 5.5: Loo loo hand rankings [108].

Five of a kind
Skip flush
Straight flush
Four of a kind
Full house
Flush
Tilter
Skip straight
Straight
Three of a kind
Blaze
Two pairs
One pair
High card

The blaze and the skip straight are hands we have seen. The following hands are new.

- A *skip flush* is a skip straight in which all 5 cards are the same suit, such as $K\heartsuit\ J\heartsuit\ 9\heartsuit\ 7\heartsuit\ 5\heartsuit$.
- A *tilter* is a hand containing a 9, a 2, and three cards between them which is not a flush. This is a variation on the skeet that need not include a 5.

Lurking outside the hands listed in Table 5.5 was the *loo loo*, from which the game took its name. A player dealt poor cards might pursue a loo loo by discarding all of his cards and drawing 6 cards to replace them. A loo loo was simply a hand containing 3 pairs. Game lore held that

> Should the player be lucky enough to bring about such a combination in the six-card draw, all that he has to do is to send his loo loo to the nearest bank—savings, national, or state—where it will be found as good as a government bond for obtaining money from the bank vaults [108].

While finding a bank willing to participate in this exchange seems unlikely, one wonders what the probability of a loo loo is. We shall make the following assumptions to guide our calculation:

1. A player pursuing a loo loo will discard 5 cards of different ranks and different suits. This means that in drawing 6 cards, 5 ranks will contain 3 cards each and 8 ranks will have the full set of 4 cards.
2. A player will not discard all of his cards and pursue a loo loo if dealt the joker.
3. Four of a kind does not count as two pairs for the purpose of constructing a loo loo.

Consider first the case where the resulting loo loo does not contain the joker. Assume that k pairs are chosen from the 8 ranks not represented among the discards, leaving $3-k$ pairs to be selected from the 5 depleted ranks. The probability of a loo loo that fits this description is

$$\frac{\binom{8}{k} \cdot \binom{4}{2}^k \cdot \binom{5}{3-k} \cdot \binom{3}{2}^{3-k}}{\binom{48}{6}}.$$

We sum this from $k = 0$ to $k = 3$ to cover all possibilities:

$$P(\text{Joker-free loo loo}) = \sum_{k=0}^{3} \frac{\binom{8}{k} \cdot \binom{4}{2}^k \cdot \binom{5}{3-k} \cdot \binom{3}{2}^{3-k}}{\binom{48}{6}} \approx .0026.$$

If the loo loo includes the joker, the hand looks like $xxyyzW$. Each of x, y, and z can be chosen from the 8 full ranks or the 5 depleted ranks. There are 6 different possibilities to consider. These are listed in Table 5.6, where "8" denotes that the variable is taken from the set of full ranks and "5" indicates a variable from a depleted rank.

TABLE 5.6: Distribution of cards among a loo loo with a joker.

x	y	z
8	8	8
8	8	5
8	5	8
8	5	5
5	5	8
5	5	5

By symmetry, the cases “858” and “588” are the same, as are “855” and “585”. If all 3 ranks are from the 8 full ranks, the probability of a loo loo is

$$\frac{\binom{8}{2} \cdot \binom{4}{2}^2 \cdot \binom{6}{1} \cdot \binom{4}{1} \cdot 1}{\binom{48}{6}} \approx \frac{1}{507}.$$

Adding up over all 6 cases gives

$$P(\text{Loo loo with a joker}) = \frac{71,802}{\binom{48}{6}} \approx .0059,$$

and adding this to the chance of a loo loo without a joker calculated above gives

$$P(\text{Loo loo}) \approx \frac{1}{118}.$$

If players show reasonable restraint in drawing for loo loos, the observation that “It is possible, but highly improbable that more than one loo loo can be played in an evening where the game is strictly on the square.” is a sensible one, although press accounts of this game give no guidance to the player wishing to approach a bank to cash in on a loo loo [108].

5.5 Alternate Decks

> *Last night I stayed up late playing poker with tarot cards. I got a full house and four people died.*
>
> —Steven Wright.

Changing the deck in poker isn’t generally lethal; it can open up some interesting mathematics that should affect gameplay. Short Deck (page 123) is perhaps the most successful poker game played with a deck containing other than 52 cards. Other decks have been proposed, manufactured, and marketed without making much of an impression on the poker community. Changing the deck can, as we have seen in considering Short Deck, lead to necessary changes in how hands are ranked, which may affect player choices. We consider the possibilities posed by changing the deck in this section.

55 Cards: Colors Can Replace Suits

In 2014, Full Color Games began marketing Full Color Cards: a 55-card deck together with new rules intended to enliven old card games, including blackjack and baccarat. This new deck consisted of aces through 11s in five colors: green, orange, purple, blue, and white. Figure 5.2 shows a straight in Full Color Cards.

FIGURE 5.2: A straight with Full Color Cards.

How does this deck change hand rankings in five-card poker?

With no face cards, royal flushes are impossible, and they are replaced as the best possible hand by five of a kind. There are 11 of these in a 55-card deck, making the probability of five of a kind

$$P(\text{Five of a kind}) = \frac{11}{\binom{55}{5}} = \frac{11}{3,478,761} = \frac{1}{316,251}.$$

Five of a kinds in 55-card deck are thus about twice as likely as royal flushes in a 52-card deck. Of course, neither hand has all that high a probability.

Straight flushes can be counted just as they are in a standard deck, by focusing on the lowest-ranked card in the straight flush. One important difference in this deck is that there are no aces that can rank both high and low—the "1" cards count only as low for poker. Any card from ace through 7 can anchor a straight flush; there are 35 of these. Once the lowest card is chosen, the other 4 cards are determined, and so the probability of a straight flush is

$$P(\text{Straight flush}) = \frac{35}{3,478,761},$$

almost exactly 1 chance in 100,000.

Continuing down through the list of standard poker hands reveals several differences from poker with a standard deck. For example, flushes are far less common with the 55-card deck than the standard deck, and four-of-a-kinds are more common. A moment's thought makes this reasonable: in moving from 4 suits of 13 cards each to 5 suits of 11 cards, we are diminishing the number of cards in any one suit by 2 and increasing by 1 the number of cards in the deck of a given rank.

Flushes are counted very simply: There are 5 suits, and 11 cards in each suit from which 5 are to be chosen. Subtracting the 35 straight flushes gives a total of

$$5 \cdot \binom{11}{5} - 35 = 2275$$

flushes.

For four-of-a-kind hands, we begin by choosing the rank from the 11 available, and then choose 4 of the 5 cards of that rank. The fifth, nonmatching card may then be any of the 50 remaining cards in the deck of other ranks, giving a total of

$$11 \cdot \binom{5}{4} \cdot 50 = 2750$$

four of a kinds.

The frequencies of the remaining 5-card poker hands in a 55-card-deck are compiled in Table 5.7.

TABLE 5.7: Poker hand frequencies: 55-card deck.

Hand	**Frequency**
Five of a kind	10
Straight flush	35
Flush	2275
Four of a kind	2750
Full house	11,000
Straight	21,840
Three of a kind	123,750
Two pairs	247,500
High card	1,419,600
One pair	1,650,000

That flushes outrank straights in the 55-card deck is no surprise, as we have seen that flushes outrank straights in a standard deck. Two surprises emerge from close inspection of this table:

- Flushes are nearly 5 times rarer than full houses. In a standard deck, full houses are less common than flushes and so outrank them.

- The most likely hand rank is one pair; high-card hands are less common than one-pair hands.

A new possibility brought about by the fifth color is a hand where all five color are represented—this is called a *rainbow* hand. Quick counting will show that there are $11^5 = 161,051$ possible rainbow hands. Five of a kind is trivially a rainbow hand, while flushes and straight flushes cannot be rainbow hands.

Example 5.5. Rainbow straights may represent a new type of hand worth enumerating. As with straight flushes, we can count rainbow straights by starting with the lowest-valued card, and again this is any card from ace through 7: 35 in all. For example, suppose that this card is the green 4. There are then four choices for the second card in this rainbow straight: all four 5s of other colors. Once this card is chosen, another color has been eliminated and only three choices remain for the third card, the 6. Following this pattern to the last card, we see that the total number of rainbow straights is

$$35 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 840.$$

■

If rainbow straights are separated out as a new kind of hand, reference to Table 5.7 shows that a rainbow straight would properly beat a flush but lose to a straight flush, and that removing these 840 hands from the roster of straights would not change the ranking of a straight relative to the other hands.

60 Cards, With A Twist: California Hold'em Revisited

While the 60-card California hold'em deck (page 127) was intended for 7-card hold'em hands, we can consider the possibilities when dealing standard 5-card hands from this deck. The standard formula with 15 ranks and 4 suits suffices for counting all hands except the various types of flushes, which require special handling to account for the unsuited 10s and 11s. The master formula can then be modified to count other hands, to

$$\binom{15}{h}\binom{4}{i}^h \cdot \binom{15-h}{j}\binom{4}{k}^j \cdot \binom{15-h-j}{m-hi-jk}\binom{4}{1}^{m-hi-jk}.$$

Since a royal flush includes a 12 instead of a 10, the number of royal flushes remains fixed at 4. Straight non-royal flushes can only have the ace through 5 as the lowest card, and so there are only 20 straight flushes, close to half the 36 in a standard deck.

Flushes and straights then take the 24 royal and straight flushes into account. There are

$$4 \cdot \binom{13}{5} - 24 = 5124$$

flushes, only slightly more than the 5108 in a 52-card deck. Full houses, which number 5040 in this deck, still beat flushes.

Straights are less restricted, and can begin with any card from ace through 12 as the lowest. There are

$$12 \cdot 4^5 - 24 = 12,264$$

straights.

Table 5.8 shows the number of each standard hand when dealt from a California hold'em deck.

TABLE 5.8: Poker hand frequencies: 60-card California hold'em deck.

Hand	**Frequency**
Royal flush	4
Straight flush	20
Four of a kind	840
Full house	5040
Flush	5124
Straight	12,264
Three of a kind	87,360
Two pairs	196,560
One pair	2,096,640
High card	3,057,660

65 Cards: Enter The Eagle

The United States Playing Card Company marketed a 5-suit deck in the 1930s. The fifth suit was crowns in England and eagles in the USA; the eagle suit was colored green [99]. While primarily intended for bridge players, who were dealt 16-card hands while the 65th card was placed face-up in the center of the table and available to the declarer, to replace a card in her hand or the dummy hand, there was some interest in 5-card and 6-card poker played with these new cards.

5-card poker hands merely require some recalculation to rank the standard hands in a 5-suit deck. Five of a kind becomes a possible hand, but it is beaten by a royal flush, while flushes outrank full houses. The master formula defined on page 22 for counting hand types can be modified for this deck by replacing the 4s, which count suits, by 5s:

$$\binom{13}{h}\binom{5}{i}^h \cdot \binom{13-h}{j}\binom{5}{k}^j \cdot \binom{13-h-j}{m-hi-jk}\binom{5}{1}^{m-hi-jk}.$$

Similarly, the general formulas that count straights, flushes, and straight flushes change, to

$$\begin{aligned} \text{Straight flushes:} \quad & 5(15-m) \\ \text{Flushes:} \quad & 5\binom{13}{m} - 5(15-m) \\ \text{Straights:} \quad & (15-m)\binom{5}{1}^m - 5(15-m). \end{aligned}$$

Table 5.9 shows the number of hands of each type in the 5-suit deck.

If poker is played with 6-card hands drawn from this deck, there are

$$\binom{65}{6} = 82,598,880$$

TABLE 5.9: 5-card poker hand frequencies: 5-suit deck [99].

Hand	**Frequency**
Royal flush	5
Five of a kind	13
Straight flush	45
Four of a kind	3900
Flush	6385
Full house	15,600
Straight	31,200
Three of a kind	214,500
Two pairs	429,000
One pair	3,575,000
High card	3,984,240

hands possible and several other new hands to consider. Flushes and straights now consist of 6 cards, suited or in sequence—or both in the case of straight flushes. The following additional hands were also recognized:

- Four of a kind plus a pair, referred to as "four and two".
- Two three-of-a-kinds, a "three and three".
- Three pairs—which was not called a "two and two and two".

Establishing the relative rankings of these new hands among the standard recognized hands is a simple matter of generalizing our earlier work with $m = 6$ in the master formula. There are

$$\binom{13}{3} \cdot \binom{4}{2}^3 = 286,000$$

ways to draw 3 pairs.

A three and three may be drawn in

$$\binom{13}{2}\binom{5}{3}^2 = 7800$$

ways. For the four and two, there are

$$\binom{13}{1}\binom{5}{4}^1 \cdot \binom{12}{1}\binom{5}{2}^1 = 7800$$

hands.

Since these last two hand types are equally likely, gameplay would require some sort of rule designed to break ties between these two hand types. One simple rule would be "Compare the rank of the 4 of a kind to the higher rank of the two 3 of a kinds. The higher rank wins". Since there are only 5 of each rank in the deck, it is impossible for this rule to result in a tie.

However, this rule slightly favors the 3 and 3 over the 4 and 2: of the 144 possible pairs of ranks, 3 and 3 wins 78 of them while 4 and 2 only wins 66. This can be corrected by looking at the problem case: where a 4 and 2 of the form $2222xx$ is involved. Since it is impossible for any 3 and 3 to have 3 deuces as its top three of a kind, all 3 and 3s will beat this hand.

One possible amendment to the tiebreaker rule might be to declare, when $2222xx$ is in a showdown with $yyyzzz$, that the rank of the pair and the lower 3 of a kind are compared, but this introduces the possibility of a tie, as when 222299 is compared to $QQQ999$.

A simple, albeit somewhat arbitrary, solution relies entirely on the 3 and 3 hand: In a showdown against $2222xx$, a 3 and 3 wins if it is at least $JJJ555$. This encompasses the top 50% of all 3 and 3 hands. The optics may occasionally be a bit strange, as when 222233 beats $TTT999$, but perhaps we can draw some comfort that the chance of one player holding a 4 and 2 while another holds a 3 and 3 is approximately 1 in 86,452,841—so this is not likely to be a frequent problem.

Table 5.10 collects the frequencies of the 14 possible 6-card hands. Note that a straight beats a full house, and a flush beats 4 of a kind, when using the 65-card deck and playing with 6-card hands.

TABLE 5.10: Six-card poker hand frequencies: 5-suit deck [99].

Hand	**Frequency**
Royal flush	5
Straight flush	40
Five of a kind	780
Four and two	7800
Three and three	7800
Flush	8535
Four of a kind	107,250
Straight	140,580
Three pairs	286,000
Full house	858,000
Three of a kind	3,575,000
Two pairs	10,725,000
High card	26,663,340
One pair	40,218,750

Another suggestion for the 65-card deck was to allow the eagle cards to be wild for suits only, thus making flushes more likely. For 5-card straight flushes, there are

- 9 ways to pick the lowest-ranked card.

- 4 choices for the red or black suit.
- 2 choices for the suit of each card: either eagles or the suit chosen above.

This gives $9 \cdot 4 \cdot 2^5$ straight flushes, but the 9 all-eagle straight flushes have been overcounted 3 times each. Subtracting 27 gives a total of 1125 5-card straight flushes. For 6-card hands, the factor of 9 above is replaced by an 8, since a 6-card 9-low straight flush is a 6-card royal flush. We then have $8 \cdot 4 \cdot 2^6 - 3 \cdot 8 = 2024$ straight flushes with 6 cards.

78 Cards: An International Deck

In 1895, Hiram Jones introduced *International Playing Cards*: a 78-card deck with 13 cards in each of 6 suits [79]. The 2 new suits were bullets (●) and crosses (✠). Originally, bullets were black and crosses were red; in more modern versions of this deck, both suits are colored blue. The International deck permits

$$\binom{78}{5} = 21,111,090$$

different 5-card hands. This number includes every hand possible in a standard deck, but the chance of a 5-card hand with no bullets or crosses is low, only

$$\frac{\binom{52}{5}}{\binom{78}{5}} \approx .1231.$$

With 6 suits, 5 of a kind is a new hand type, but ranking it among other hands is not a simple matter of assigning it the highest rank, as we saw when considering the Eagle deck. Royal flushes and straight flushes both outrank 5 of a kind.

- Royal flushes are easiest to count: there's one for each suit, including ●s and ✠s, for a total of 6.
- Straight flushes are also easy: $6 \cdot 9 = 54$.
- Five of a kinds number $13 \cdot \binom{6}{5} = 78$, making them the third-ranked hand.

Example 5.6. Assuming a modern International deck with blue ●s and ✠s, the chance of a 5-card hand where all of the cards are the same color is

$$\frac{3 \cdot \binom{26}{5}}{\binom{78}{5}} \approx .0093$$

—less than 1% of all 5-card hands. ■

One-color hands were not entirely new with the International deck; they were mentioned in 1880 by John Blackbridge, who asserted that a hand containing 5 cards of the same color—if recognized as distinct—should rank between 1 pair and 2 pairs [8]. Exercise 5.11 considers what might happen if one-color hands were singled out as a separate hand type in a standard deck.

Rainbow hands seem like they should be easier to draw in an International deck, since there are 6 suits from which to choose 5. There are

$$\binom{6}{5} \cdot 13^5 = 2,227,758$$

possible rainbow hands, so the probability of a 5-card hand with 5 different suits present is

$$\frac{2,227,758}{21,111,090} \approx .1055.$$

This is over double the probability of a rainbow hand in a 55-card deck computed above.

If we consider 6-card rainbow hands, they number $13^6 = 4,826,809$—more than double the number of 5-card rainbow hands, but with a lower probability: .0188.

5.6 Deck Composition More Generally

As we have seen, small changes in deck composition can have a big effect on hand rankings. Let's take a look at some more general questions about card decks, independent of whether or not the deck we choose has ever been manufactured.

Consider a deck with r ranks and s suits, for a total of rs cards. The master formula described on page 22 can then be generalized to count 5-card hands. The number of ways to draw exactly h sets of i cards of the same rank and j sets of k cards of the same rank among a hand of $m = 5$ cards, where $hi + jk \leqslant 5$, from this deck is

$$\binom{r}{h}\binom{s}{i}^h \cdot \binom{r-h}{j}\binom{s}{k}^j \cdot \binom{r-h-j}{5-hi-jk}\binom{s}{1}^{5-hi-jk}.$$

Similar modifications may be made to count straights, flushes, and straight flushes in this deck.

$$\begin{aligned} \text{Straight flushes:}\quad & s(r-3) \\ \text{Flushes:}\quad & s\binom{r}{5} - s(r-3) \\ \text{Straights:}\quad & (r-3)\binom{s}{1}^5 - s(r-3). \end{aligned}$$

We assume here that the deck contains aces which can count both high and low in a straight or straight flush.

Full Houses vs. Flushes

For full houses and flushes, we require $r \geqslant 5$ and $s \geqslant 3$. The number of flushes, with straight flushes excluded, is

$$s \cdot \binom{r}{5} - s(r-3) = \frac{s}{120} \cdot (r-3)(r^4 - 7r^3 + 14r^2 - 8r - 120),$$

and the number of full houses is

$$r \cdot \binom{s}{3} \cdot (r-1)\binom{s}{2} = \frac{s^2}{12} \cdot r(r-1)(s-2)(s-1)^2.$$

For a fixed value of s, flushes outrank full houses when r is small, but full houses eventually become scarcer than flushes. In the standard 52-card deck, we have $r = 13, s = 4$, and full houses rank higher. This is the lowest value of r in a 4-suit deck where a full house beats a flush; in a 12-rank, 4-suit deck such as is used in the blackjack variant Spanish 21 (all of the 10s, but no face cards, are removed), there are 3132 flushes and 3168 full houses, making flushes slightly rarer.

At the same time, fixing r and allowing s to increase reveals a point where full houses become more numerous than flushes. Again, the standard deck sits at a transition point, this time involving s. In a 5-suit deck with 13 ranks, there are 6385 flushes and 15,600 full houses, and so flushes properly outrank full houses. In a deck with only 3 suits, flushes rank higher if there are 6–8 ranks. (In a 5-rank deck, all flushes are straight flushes, and so the number of "proper" flushes is 0.) With 9 or more ranks, flushes are more numerous than full houses.

Straights vs. Flushes

If we count the number of straights in a general deck, we find

$$s(r-3)(s^4-1)$$

straights. Straights outrank flushes if

$$s(r-3)(s^4-1) < \frac{s}{120} \cdot (r-3)(r^4 - 7r^3 + 14r^2 - 8r - 120),$$

or

$$s < \sqrt[4]{\frac{1}{120}(r^4 - 7r^3 + 14r^2 - 8r - 120) + 1}.$$

For ease of calculation, the right-hand side of this inequality can be approximated very accurately, for $4 \leqslant r \leqslant 20$, by a simple linear function:

$$s < .3195r - .8195 \approx \frac{r}{3} - 1.$$

Table 5.11 shows the minimum value of r in a deck with s suits where straights are rarer than flushes.

TABLE 5.11: Deck parameters where straights outrank flushes.

Suits, s	**Minimum ranks, r**
3	12
4	16
5	19
6	22
7	25

Straights vs. Full Houses

If there are 10 or fewer suits, straights only outrank full houses if

$$s(r-3)(s^4-1) < \frac{s^2}{12} \cdot r(r-1)(s-2)(s-1)^2.$$

Evaluating this inequality for integer values of r and s shows that straights are rarer only if there are at least 17 ranks and $r \geqslant 27 - s$. The minimum deck size arises with a 24-rank, 6-suit deck, which has 144 cards, nearly 3 standard decks.

For each of the 6 possible orderings of full house, flush, and straight, Table 5.12 shows a deck configuration in which the hands are ranked in that order. The size of the deck rises when straights are required to outrank full houses, and it is considerably harder for straights to outrank both full houses and flushes in a comfortably small deck.

TABLE 5.12: Deck sizes for all rankings of full house, flush, and straight.

Order	r	s	**Deck size**	**Description**
$FH > F > S$	13	4	52	Standard deck
$FH > S > F$	16	4	64	Lucky 13s deck (p. 179)
$F > FH > S$	11	5	55	55-card deck (p. 169)
$F > S > FH$	21	7	147	Minimum deck size
$S > F > FH$	25	7	175	Minimum deck size
$S > FH > F$	24	6	144	Minimum deck size

A 64-card deck where full houses outrank straights, which in turn outrank flushes, was used in an Australian blackjack variant called *Lucky 13s Blackjack*. This deck added 11s, 12s, and 13s in all 4 suits to the standard deck, giving a deck with 16 ranks and 4 suits [10].

If you're playing poker and are dealt a 21 of hearts, as was Bugs Bunny while playing blackjack in the 1959 cartoon *Bonanza Bunny*, be aware that you're playing with a deck where a straight beats a full house—and it's possible that a flush also beats one or both of them.

We note that the standard deck configuration sits at a transition point; almost any change in the number of suits or ranks will change the order of these three hands. Table 5.13 shows the effect of changing the number of suits or ranks by 1.

TABLE 5.13: Effect of changing the number of suits or ranks on the relative order of full houses (FH), flushes (F), and straights (S). The standard 52-card deck is highlighted.

r \ s	**3**	**4**	**5**
12	$FH > S > F$	$F > FH > S$	$F > FH > S$
13	$FH > S > F$	$\boldsymbol{FH > F > S}$	$F > FH > S$
14	$FH > S > F$	$FH > F > S$	$F > FH > S$

Changing even the number of cards by 1 can make a difference in hand rankings. The 64-card Lucky 13s blackjack deck has only 1 fewer card than the Eagle deck, but the distribution between ranks and suits means that there are some differences in the rank of 5-card hands. The Eagle deck admits 5-of-a-kind hands that are absent in Lucky 13s; a further difference occurs when ranking straights and flushes. In a 64-card deck with 16 ranks and 4 suits, straights outrank flushes 13,260 to 17,240. If we add a card and redistribute to a deck with 13 ranks and 5 suits, flushes are rarer by a count of 15,600 to 31,200. Full houses outrank both flushes and straights in either deck.

Example 5.7. The United States Playing Card Company published a guide to poker in 1941 [151]. This book suggested that, in a small poker game, the 2s and 3s be removed from the deck, creating a deck with 44 cards. In this deck, flushes are scarcer than full houses, which in turn outrank straights. For games with more than 8 players, a 60-card deck including 11s and 12s in all 4 suits was recommended. This deck ranks full houses, flushes, and straights in the same order as in the 52-card deck. ■

These multivariable inequalities can be used to examine deck configuration as it affects the ranks of other hand types. We might, ask, for example, if it's possible to have a deck where 4 of a kind beats a straight flush.

At first glance, building this deck seems like a simple matter of having lots of ranks but few—perhaps the bare minimum of 4—suits, thus giving limited ways to draw 4 cards of the same rank. What makes this idea more complicated is the fact that the fifth card to 4 of a kind can be any card in the deck, and if rs is large, this gives many possibilities.

For four-of-a-kind hands to outrank straight flushes, we need

$$\binom{r}{1}\binom{s}{4} \cdot (r-1)(s) < s(r-3),$$

where $r \geqslant 5$ and $s \geqslant 4$. Again, we assume that the deck includes aces which can be counted as both the high or low card in a straight flush. Expanding the combinations gives

$$\frac{r \cdot s(s-1)(s-2)(s-3)}{24} \cdot (r-1)s < s(r-3).$$

Consider the smallest viable case: $s = 4$. We then have

$$r(r-1) < r-3,$$

or

$$r^2 - 2r + 3 < 0,$$

which has no real solutions.

Proceeding to the general case, we can rearrange the inequality above to yield

$$\frac{r \cdot (s-1)(s-2)(s-3)}{24} \cdot (r-1) - (r-3) < 0,$$

where we may safely divide out a factor of s since $s > 0$. Figure 5.3 shows a 3-dimensional plot of the left side of this inequality for $5 \leqslant r \leqslant 25, 4 \leqslant s \leqslant 25$. We note that this is never negative and is increasing as both r and s increase,

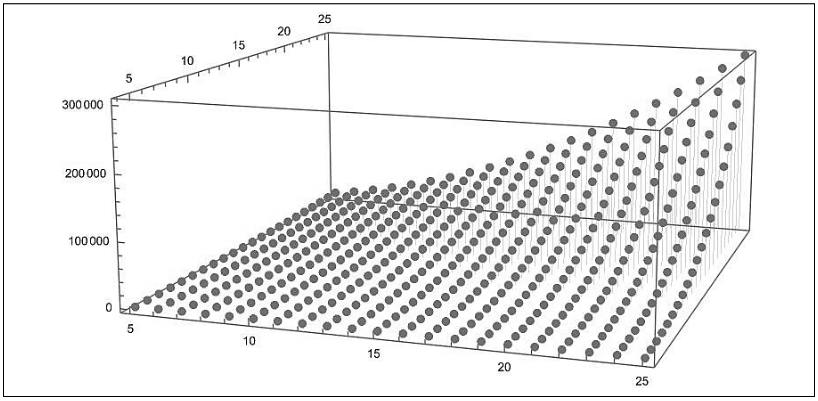

FIGURE 5.3: Mathematica plot of $\frac{r \cdot (s-1)(s-2)(s-3)}{24} \cdot (r-1) - (r-3)$ over the intervals $5 \leqslant r \leqslant 25, 4 \leqslant s \leqslant 25$.

meaning that it is impossible to construct a deck where 5-card straight flushes are more numerous than 4-of-a-kind hands. Table 5.14 collects some values of this expression for small values of r and s.

TABLE 5.14: Numerical values of $\dfrac{r \cdot (s-1)(s-2)(s-3)}{24} \cdot (r-1) - (r-3)$.

r \ s	**4**	**5**	**6**	**7**	**8**
5	3	18	48	98	173
6	4.5	27	72	147	259.5
7	6.5	38	101	206	363.5
8	9	51	135	275	485
9	12	66	174	354	624
10	15.5	83	218	443	780.5
11	19.5	102	267	542	954.5
12	24	123	321	651	1146
13	29	146	380	770	1355

It can be seen that $\dfrac{r \cdot (s-1)(s-2)(s-3)}{24} \cdot (r-1) - (r-3)$ is increasing when either r and s increases, and so does not ever give a negative number. (An observer might be concerned by the fractional values in Table 5.14 for some even values of s. When we simplified the expression by dividing out an s, the resulting simpler formula does not necessarily correspond any longer to something that can be counted, hence some non-integer values are possible.)

This assumes, however, that each card is represented only once in the deck. Recall that pinochle uses a 48-card deck consisting of 2 copies each of the 9s through aces in all 4 suits. Perhaps duplication of one or more cards will change things.

Suppose that there are c copies of each combination of suit and rank in a deck. The full deck contains crs cards, and the general formula for counting 5-card hands on page 22 can be further generalized to

$$\binom{r}{h}\binom{sc}{i}^h\binom{r-h}{j}\binom{sc}{k}^j\binom{r-h-j}{5-hi-jk}\binom{sc}{1}^{5-hi-jk}.$$

Example 5.8. In a pinochle deck, we have $r = 6,\ s = 4,\ c = 2$, and thus there are 47,040 full houses. The probability of a full house is about 2.75%, considerably greater than the 0.14% probability of a full house from a standard deck.

Five of a kind is a possible hand from this deck, and can be dealt in

$$\binom{6}{1}\binom{8}{5}^1 = 336$$

ways. The chance of 5 of a kind is about $\dfrac{1}{5096}$. ■

Returning to straight flushes and four-of-a-kinds, we are faced with the following inequality if 4 of a kind is to beat a straight flush:

$$\frac{r(sc)(sc-1)(sc-2)(sc-3)}{24} \cdot (r-1)(sc) < s(r-3) \cdot c^5.$$

While both sides are quintic polynomials in c, the left side is also quintic in s. The right side is only linear in s, suggesting again that this may not be solvable for appropriate values of r, s, and c.

Computation of both sides of the inequality for $5 \leqslant r \leqslant 25, 4 \leqslant s \leqslant 25$, and $1 \leqslant c \leqslant 10$, ranges which cover deck sizes as large as 6250 cards, shows no case where 4 of a kind outranks a straight flush. For a 6250-card deck, the left side exceeds the right side by over 23 trillion. That deck would be 1.92 meters thick, and certainly a challenge for the dealer to handle.

5.7 Exercises

Answers to starred exercises begin on page 338.

5.1.* The following problem is adapted from [27]. In a game of straight poker (page 165), the hand is interrupted after 3 cards have been dealt, with Terry holding $QT7$ of diamonds and Robin holding $A\spadesuit$ $A\diamondsuit$ $A\clubsuit$. Regarding each player's hand as a completed 3-card hand, who should be declared the winner?

Nonstandard Hands

5.2. Confirm by direct calculation that there are 192 blazes which contain exactly three of a kind.

5.3.* An 1891 book by W.J. Florence, written, it is said, in less than four weeks, gives an alternate definition for a tiger as the lowest possible hand: an unsuited 75432 [39]. Find the probability of this kind of tiger and determine where it ranks among the 10 standard poker hands.

5.4.* In Exercise 3.11, we considered Woolworth Draw. The Woolworth chain of discount stores was also immortalized in poker with the nonstandard hand called the *Woolworth*. This hand consists of a 10, a 5, and 3 cards of different ranks in between them, and beats 3 of a kind while losing to a straight [115]. How many Woolworths, with flushes excluded, are possible?

5.5.* A nonstandard poker hand that requires a deck with at least 1 wild joker is the *flash*, which is any 5-card hand containing a joker and 1 card from each suit [115]. The flash is similar to a rainbow hand (page 171). As with blazes and rainbow hands, a hand can qualify as a flash, when recognized, and not be counted as a flash if it ranks higher. One such hand would be the full house $K♠\ K♢\ 4♡\ 4♣\ W$. Since a flash typically ranks between 2 pairs and 3 of a kind, this upward valuation is not uncommon. If the joker is fully wild, the lowest flash is a pair.

In a 53-card deck with 1 joker, find the number of hands that qualify as a flash, irrespective of whether or not they're counted that way.

5.6.* The hands that play as flashes must all have the form $Wabcd$, where W represents the joker and a, b, c, and d stand for different ranks. A hand of the form $Waabc$ will be played as 3 of a kind, not a flash. In a 53-card deck with 1 wild joker, how many flashes have the form $Wabcd$?

5.7.* Some of the flashes in Exercise 5.6 will be played as straights, even if the joker is restricted to use as a bug. Find the probability of a flash that forms a straight.

5.8. Which should rank higher, a flash or a blaze?

5.9.* Most descriptions of skip straights describe them as hands where the card ranks are separated by exactly 2. How many skip straights with ranks separated by 3 are possible? Exclude skip straight flushes from this calculation.

5.10. Show that, if they are separated out as a distinct hand type, a round-the-corner straight (page 4) outranks a full house.

5.11. Following up on Example 5.6 (page 176), there are $2 \cdot \binom{26}{5} = 131,560$ one-color hands in a standard deck, slightly more than the number of two-pair hands. Of course, many of these rank higher than 2 pairs—a straight flush or royal flush, for example, is necessarily a one-color hand. Three-of-a-kind hands cannot be one-color hands.

Find the probability of a hand containing 5 cards of the same color and ranking 1 pair or lower. Is John Blackbridge's assertion that a 1-color hand should rank between 1 pair and 2 pairs correct?

5.12. How does the probability of a loo loo drawn from a fresh 53-card deck compare to the probability calculated on page 169?

Alternate Decks

5.13.* Templar, a 19th-century poker writer, recommends that, in a 2- or 3-handed poker game, using a 32-card or "piquet" deck enlivens the game [147]. This deck consists of the 7s through aces in all 4 suits. How many full houses are possible with this deck?

5.14. In a piquet deck, which rank higher: straights or flushes? In counting straights, the ace is regarded as either high or low, so $A789T$ is a possible straight.

5.15. Show that in a pinochle deck (page 182), straight flushes and non-straight flushes are equally likely. Assume that the ace counts both high and low, so $A9TJQ$ is a straight flush.

5.16.* In a game of five-card poker with a 55-card deck, find the number of rainbow poker hands with the following ranks:

a. Four of a kind

b. Full house

c. Two pairs

d. High card

5.17. Confirm by direct calculation that the number of full houses in a 55-card deck is 11,000.

5.18.* The equivalent of a rainbow hand in California hold'em would be a 5-card hand with one card from each suit and one unsuited 10 or 11. How many of these hands are possible?

5.19. One drawback to 6-card poker played with the 5-suit Eagle deck is that a hand like 6♣ 5♣ 4♣ 3♣ 2♣ J♠ contains a 5-card straight flush, but counts only as a high-card hand if Table 5.10 is used to rank hands. Some gaming authorities advocated for 5-card straights, flushes, and straight flushes to count as distinct hand types and to be ranked accordingly [99]. Find the probability of a 6-card hand containing a 5-card straight flush.

5.20.* Consider a poker game where 5-card hands are dealt from a pack of 3 standard decks of cards. A possible new hand using this deck is a *full flush*: a full house with all 5 cards of the same suit. Find the probability of a full flush.

Chapter 6

Beyond Hold'em

Games in which players face off directly and compete against each other for the best hand, many of which are played in California cardrooms, are the focus of this chapter. For over 100 years, California law permitted draw poker games in the state's card rooms, but "stud-horse" poker was specifically prohibited, having been banned by the state legislature in 1885. Exactly what was meant by the nonstandard term "stud-horse" poker was a matter of some speculation among poker players; news accounts of the period describe a game very much like 5-card stud [54, 146]. This interpretation was finally confirmed in 1947, when the California attorney general declared that stud-horse poker was the same thing as stud poker [116]. While this may have clarified the issue somewhat, the ruling was challenged by legal scholars who claimed that the legislature's intention was to ban house-banked games, not stud poker *per se*. This confusion persisted in the subsequent years.

In the interim, and continuing after the 1947 declaration, a variety of games based on the same general principles as draw or stud poker have been invented and found their way to card rooms in California, and some have been picked up elsewhere. An excellent Internet resource for information on games played in California card rooms is the California Gaming Establishments List maintained by the Attorney General's office. Its Web site is `https://oag.ca.gov/gambling/cardroomlist`; links to each card room's game rules may be found there.

6.1 Lowball Games

Archie

Deck composition:	52 cards
Hand size:	5 cards
Community cards:	0
Betting rounds:	4

At the Bicycle Hotel and Casino in Bell Gardens, California, *Archie* is a combined high/low game involving 4 betting rounds and 3 draw opportunities.

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Since players may discard all 5 of their cards at any draw opportunity without folding, it is quite possible that the deck may be exhausted by all of the drawing. There are specified rules for shuffling discards from an earlier round for use in later rounds.

The final pot is split evenly between the highest hand and the lowest, with the following provisions:

- The high hand must be at least a pair of 6s.
- The low hand cannot include cards higher than a 6, with aces counting low, so a 6-high hand is the highest possible low hand.
- Straights and flushes do not count in the low hand. A 6- or 5-high straight might be a strong contender to take the whole pot by counting both high and low.

  There are 1024 straights, including straight flushes, of the form 65432, and 1024 possible wheels ($5432A$). There are 4 possible forms for a 6-high flush that is not also a straight: $6543A, 6542A, 6532A$, and $6432A$. Each of these hands can occur in 4 different suits; adding everything up gives 2064 hands that have a good shot at winning both high and low.

Example 6.1. Suppose that a player is dealt $T\heartsuit$ 8♠ 5$\diamondsuit$ 2$\diamondsuit$ A♣. As this is not promising as a high hand, the player opts to discard the 10 and 8 and pursue a low hand. If the cards received in exchange are 6$\heartsuit$ 2♣, the hand is now a pair of 2s. This hand cannot qualify as either high or low, so the player would likely discard a deuce. If the replacement card is the 3$\diamondsuit$, the hand is now 6-high, and thus a strong contender to win the low half of the pot. Standing pat on the third draw is the best decision. ■

The criterion for low hands seems restrictive, even allowing for the three chances to improve a hand by discarding and drawing. How many 6-high or lower hands are possible?

This number, 6144, can be obtained by adding the numbers of 6-high and 5-high hands from Table 1.3. We can also compute it directly: Since flushes and straights do not count against a low hand, there are

$$\binom{6}{5} \cdot 4^5 = 6144$$

possible low hands. The $\binom{6}{5}$ factor selects 5 ranks from 6 down to ace; there are then 4 ways to pick each card once its rank is set. Approximately 1 hand in 423 is low enough to count as low in Archie.

Example 6.2. In the final Archie showdown, if no hand qualifies as either low or high, the hand is awarded to the best nonqualifying high hand. How many hands fail to qualify as both high and low?

We want to count the number of hands between a pair of 5s and a 7-high hand, with straights and flushes excluded since those are qualifying high hands. Table 5.2 (page 158) makes the counting easy. There are

$$4 \cdot \frac{1,098,240}{13} = 337,920$$

one-pair hands with a pair of 2s through 5s, and 1,302,540 high-card hands from 7-high though ace-high, for a total of 1,640,460 hands. ■

While the probability of a nonqualifying high hand is .6312 if we look only at dealt initial hands, the opportunity to discard and draw up to 3 times means that the likelihood that a player will approach the final showdown with one of these hands is considerably smaller.

Badugi

Deck composition:	52 cards
Hand size:	4 cards
Community cards:	0
Betting rounds:	3

Badugi (pronounced *bad′-a-gee*) is a lowball poker game that uses 4-card hands. At the Bicycle Casino, players compete against each other in a race to the lowest hand which includes 3 betting rounds and 2 opportunities to discard and draw new cards. Other poker rooms may play the game with 4 betting rounds and 3 draws. The lowest possible hand is called a badugi, and consists of 4 cards of different ranks and different suits: $J\spadesuit$ 5♡ 3♣ 2♢, for example. Recall that a collection of playing cards—a full or partial hand—where each card is a different suit may be called a *rainbow* hand, so every badugi is a rainbow hand. A badugi beats any non-badugi, so the hand listed here would beat an apparently-lower hand such as 7♡ 6♠ 5♠ 3♣. Aces always rank low, and suits do not count in ranking hands.

Four cards are dealt face down to each player, and a round of betting follows. Players who do not fold in the first round may then stand pat or may discard 1–4 of their cards and receive new ones. A second round of betting and a second chance to exchange cards follow. If the dealer should run out of fresh cards, the discards may be shuffled and recirculated.

After a final round of betting, the players who remain compare hands, with the lowest hand winning. Three types of hands are recognized; from best to worst, they are

- Badugis.
- 3-card incomplete hands, which have 2 cards either suited or paired, for example, $K\clubsuit$ 9♢ 5♡ 4♡. One card keeps this hand from being a badugi.

- 2-card incomplete hands, with 3 cards suited or paired, such as $Q\diamondsuit$ 7$\heartsuit$ 7$\diamondsuit$ 7$\clubsuit$.

Example 6.3. Suppose a player is dealt 7$\clubsuit$ $Q\clubsuit$ 5$\clubsuit$ $T\clubsuit$. While a 4-card flush is a fine start in many forms of poker, it's deadly in badugi. A player who chooses not to fold this hand might discard 3 cards, retaining only the 5. If the new cards are $Q\spadesuit$ $J\clubsuit$ 9$\heartsuit$, the hand has advanced to the 3-card incomplete hand $Q\spadesuit$ $J\clubsuit$ 9$\heartsuit$ 5$\clubsuit$. Discarding the queen and jack is the prudent move—if not folding—and if the player then draws $Q\diamondsuit$ 8$\spadesuit$, the resulting hand is a queen-high badugi.

A second player receives $K\clubsuit$ 8$\diamondsuit$ 8$\clubsuit$ $A\diamondsuit$ to start, which has a pair of 8s and two sets of 2 suited cards. She discards the $K\clubsuit$ 8$\diamondsuit$ and receives 4$\diamondsuit$ $A\spadesuit$. While aces rank low, 2 of them means that a badugi is out of the question, and so in an effort to go lower, she discards the 8$\clubsuit$ and $A\diamondsuit$. If her second draw is $J\heartsuit$ 6$\diamondsuit$, her final hand is $J\heartsuit$ 6$\diamondsuit$ 4$\diamondsuit$ $A\spadesuit$, a 3-card incomplete hand. This hand is lower than the pair of aces hand which was broken up in the second draw. ■

The probability of being dealt a badugi is computed by counting the number of ways to assign 4 different ranks to the 4 suits. Order matters in this count, since different ranks are assigned to different suits: 7$\clubsuit$ 5$\diamondsuit$ 4$\heartsuit$ 2$\spadesuit$ is different from 7$\heartsuit$ 5$\spadesuit$ 4$\clubsuit$ 2$\diamondsuit$. The probability is

$$\frac{{}_{13}P_4}{\binom{52}{4}} = \frac{17,160}{270,725} \approx .0634,$$

just over 6.25%, so badugis are not especially rare: about 1 hand in 16.

Another way to compute this probability is to go card-by-card and use conditional probability. The first card dealt to a badugi can be anything. The second card must not match the rank or suit of the first card; this event has probability $\frac{36}{51}$. Assuming a successful draw, there are 50 cards remaining, and 22 of them match neither the ranks or suits of the first two cards—11 in each of the 2 suits left undrawn. Finally, with 3 cards to a badugi, there are 10 cards in the missing suit that complete the hand. The probability of a badugi is found by multiplying these fractions:

$$\frac{52}{52} \cdot \frac{36}{51} \cdot \frac{22}{50} \cdot \frac{10}{49} = \frac{264}{4165},$$

or the same .0634 found above.

When players are given two chances to improve their hand toward a badugi and to fold instead of betting further, seeing one or more badugis in the final showdown would not be unexpected.

Badugi has been used as the jumping-off point for a variety of other poker games. We briefly examine those next.

Badugi High-Low

Deck composition:	52 cards
Hand size:	4 cards
Community cards:	0
Betting rounds:	3

The split-pot game *Badugi High-Low* is simple enough to describe: the pot is divided between the lowest and highest badugi at the showdown. The challenge come when identifying the highest badugi: is it a hand like $K♣\ 9♡\ 6♢\ 4♠$, with a high highest card, or one like $J♢\ T♠\ 9♡\ 7♣$, with a higher lowest card?

Either definition might be reasonable, but it's important that everyone involved in a game—including card room officials who might be called upon to arbitrate—agree. Badugi high-low rules decree that the highest-ranking badugi is the one with the highest low card, so $J♢\ T♠\ 9♡\ 7♣$ would outrank $K♣\ 9♡\ 6♢\ 4♠$ and the best possible badugi would be $AKQJ$ of 4 different suits. In short, badugis are ranked by their highest and lowest cards: the hand $K♣\ 9♡\ 6♢\ 4♠$ would be a "king-low, 4-high" hand. If another player holds a badugi, this hand seems unlikely to win either half of the pot.

Aces are very powerful cards in badugi high-low, since they count as both high and low cards. A player holding $A♡\ 7♢\ 6♠\ 5♣$ holds a badugi that is at once 5-high and 7-low, and so stands a good chance of scooping the pot.

Badeucy and Badacey

Deck composition:	52 cards
Hand size:	5 cards
Community cards:	0
Betting rounds:	3

Badeucy divides the pot evenly between the best deuce-to-seven low hand and the best badugi. Since the lowball side of the game is deuce-to-seven, aces are always high and straights and flushes count as high hands. Players are permitted one draw to their initial 5-card hands.

The probability of a 4-card badugi in a player's dealt 5-card hand is

$$\frac{4 \cdot \binom{13}{2} \cdot {}_{12}P_3}{\binom{52}{5}} = \frac{411,840}{2,598,960} \approx .1585.$$

Note that this formula allows for a hand that contains a pair, provided that one card of the pair is contained in the suit with 2 cards present. Roughly 16% of the time, a player can draw with the goal of lowering a hand that contains a badugi.

By contrast with badeucy, the object in *badacey* is to make the best ace-to-five lowball hand along with the best badugi. Here, straights and flushes do not count as high hands and the ace ranks low, so a wheel is the best low hand and a rainbow $432A$ is the best badugi.

Drawmaha Low-Dugi and Drawmaha High-Dugi

Deck composition:	52 cards
Hand size:	9 cards
Community cards:	5
Betting rounds:	4

Drawmaha Low-Dugi, as per the name, combines Drawmaha with badugi. As noted in Section 4.3, Drawmaha games are split-pot affairs, awarding half of the pot to the high Omaha hand and half to a hand chosen by another method. Here, that half of the pot is awarded to the player whose 4 hole cards comprise the lowest badugi. In the likely event that no one holds a badugi in the hole, the lowest 3-card badugi takes half of the pot.

Once Drawmaha Low-Dugi appears, it's a very short leap to *Drawmaha High-Dugi*. As the name suggests, half of the pot goes to the player whose hole cards form the highest badugi, or highest 3-card badugi if no one has a 4-card badugi. By contrast with Badugi High-Low, pairs, triplets, and 4-of-a-kind hands can count as high badugis, outranking hands with 4 different suits and 4 different ranks. This is one Omaha variant where holding all 4 aces in the hole is beneficial, since that is the highest possible badugi.

6.2 Division Games

Some poker games call for players to take their dealt cards and divide them into 2 or 3 smaller hands, which face off against other players' similarly-sized hands. We shall call these *division games*; they can be found in player vs. player games as well as carnival games (Section 7.3) where players compete only against the dealer's hand.

Chinese Poker

Deck composition:	52 cards
Hand size:	13 cards
Community cards:	0
Betting rounds:	1

At the Gardens Casino in Hawaiian Gardens, California, a 2–4 player division game called *Chinese Poker* or *13-Card Poker* deals 13 cards to each player. Players arrange their hand into 3 separate hands: a bottom or back hand of 5 cards, a middle hand of 5 cards, and a top or front hand of 3 cards. The names of the 3 hands refer to their placement in front of each player: the 3 hands are arranged in a column with the front hand farthest from the player and the back hand closest. The cards must be arranged so that the bottom hand beats or ties the middle hand and the middle hand beats or ties the top hand. If this is not done, the hand is foul and the player must pay a penalty to his or her opponents. Straights and flushes have no standing in the 3-card hand; the only recognized top hands are 3 of a kind, 2 pairs, and high card.

Example 6.4. Suppose the player is dealt the following cards:

$$\spadesuit A2, \heartsuit T5, \diamondsuit KQT63, \clubsuit AQ54.$$

One possible valid division of these cards into the required 3 hands would be

Bottom:	$6\diamondsuit\ 5\clubsuit\ 4\clubsuit\ 3\diamondsuit\ 2\spadesuit$	Straight
Middle:	$Q\diamondsuit\ Q\clubsuit\ T\heartsuit\ T\diamondsuit\ K\diamondsuit$	Two pairs
Top:	$A\spadesuit\ A\clubsuit\ 5\heartsuit$	Pair.

■

Once all hands are complete, players compare their bottom, middle, and top hands to the corresponding hands of each of the other players. Each hand that outranks an opponent's corresponding hand scores a point. Additional points may be awarded based on the value of the individual hands; for example, a royal flush in scores 25 points in the bottom hand and 50 points in the middle hand. At the end of the showdown, each point is valued at an agreed-upon monetary value, and players settle up, with players with fewer points paying players with more points.

For a middle hand to earn 50 points from a royal flush, the bottom hand must also hold a royal flush. The probability of holding 2 royal flushes among 13 cards is

$$\frac{\binom{4}{2}\cdot\binom{42}{3}}{\binom{52}{13}} = \frac{3}{27,657,385} \approx 1.8047\times 10^{-7}.$$

At the 101 Casino in Petaluma, California, the game is called Face-Up Chinese Poker. Only five cards are dealt initially; players must then assign those cards to the three hands. Subsequent cards are dealt and placed individually until all 3 hands are complete, which adds a different challenge to a hand of Chinese Poker. The cards are dealt and placed face-up, so players have full knowledge of the undealt cards when allocating their cards to their 3 hands.

Certain combinations of cards are designated Clean Sweeps, depending on the casino, and are awarded bonus points whose value may be collected from opponents. Across California card rooms, the following 13-card hands may be Clean Sweep hands:

- **Super Dragon** or **Pure Dragon**: Ace through king in a single suit.
- **Black** or **Red Dragon**: Ace through king of the same color.
- **Dragon**: Ace through king, any suits.
- **All Blacks** or **All Reds**: 13 cards of the same color.
- **Twelve Black** or **Twelve Red**: Twelve black cards with one red card or twelve red cards with one black card.
- **Minor Hand**: All cards are 2 through 9.
- **Six Pairs**: Six pairs, where four of a kind may be counted as two pairs.
- **Three Straights**: All three hands—back, middle, and front—contain a straight.
- **Three Flushes**: All three hands contain a flush.

Since the players settle their wins among themselves, the card room makes its money by collecting a fee from each player for access to the game and the services of a dealer. At the 101 Casino, the fee is based on the table limit (value per point) and is charged for every 40-minute gaming session. For example, at a \$1 table, players incur a \$10 charge every 40 minutes, though players who join a game after the 20-minute mark of a given time block are charged only half this assessment. If a point is valued at \$20, the fee rises to \$12 per 40 minutes.

Remember that there are $\binom{52}{13} = 635{,}013{,}559{,}600$ possible 13-card hands. This will be the denominator in the calculations that follow.

There are only 4 ways to draw a Super Dragon hand. Black and Red Dragons offer 2 choices for each card, for a total of 2^{13} hands. From this number, we subtract the 2 Super Dragons in that color, so the probability of a Black or Red Dragon is

$$\frac{2^{13}-2}{\binom{52}{13}} \approx \frac{1}{77{,}535{,}233}.$$

The Dragon hand may be drawn in $4^{13} - 4 = 67{,}108{,}860$ ways, which sounds lucrative until we compute the probability of a Dragon:

$$\frac{4^{13}-4}{\binom{52}{13}} = \frac{67{,}108{,}860}{635{,}013{,}559{,}600} \approx 1.057 \times 10^{-4},$$

or approximately once in 9462 hands.

Twelve Black and Twelve Red hands are equal in number, and can be counted without the need to account for any higher-ranked Clean Sweeps other than Dragons, since the other hands all rely on a single suit or color across all 13 cards. For either type of hand, there are

$$\binom{26}{12} \cdot 26 - (4^{13} - 4) = 183,991,340$$

ways to draw it.

Six Pair hands can be broken down by the number of four-of-a-kinds they contain. A Six Pairs hand with k four-of-a-kinds, where $0 \leqslant k \leqslant 3$, can be drawn in

$$\left[\binom{13}{k} \cdot \binom{4}{4}^k\right] \cdot \left[\binom{13-k}{6-2k} \cdot \binom{4}{2}^{6-2k}\right] \cdot 40$$

ways. The factor in the first set of brackets counts four-of-a-kinds; the second bracket counts pairs. The factor of 40 at the end represents the number of ways to draw the 13th, unmatched, card. This card can be any card still remaining in the deck, including one that matches one of the pairs.

Since $\binom{4}{4} = 1$ and $\binom{4}{2} = 6$, this can be simplified to

$$\binom{13}{k} \cdot \binom{13-k}{6-2k} \cdot 6^{6-2k} \cdot 40.$$

If a hand contains no four-of-a-kinds and is made up of six different pairs, it can occur in

$$\binom{13}{6} \cdot 6^6 \cdot 40 = 3,202,467,840$$

ways. The numbers drop off sharply as the number of four-of-a-kinds rises. If $k = 1$, we have

$$13 \cdot \binom{12}{4} \cdot 6^4 \cdot 40 = 333,590,400$$

Six Pairs hands with a single four-of-a-kind. Similarly, there are 6,177,600 hands with 2 four-of-a-kinds and 2 other pairs, and 11,440 hands with 3 four-of-a-kinds and a single unmatched card. The total number of Six Pairs hands is then 3,542,247,280, giving the probability of Six Pairs as

$$\frac{3,542,247,280}{635,013,559,600} \approx .0056,$$

just more than ½%.

6.3 Stud Games

California gaming laws have always permitted draw poker, as it is regarded as a game with a skill component: the decision of which cards to discard and replace. Stud-horse (or just stud) poker was considered to be a game of chance and as such was forbidden by California law. Accordingly, the market was receptive to nonstandard stud poker games, and several California stud variations, as well as some stud games played in other states, are considered in this section.

Hurricane

Deck composition:	52–54 cards
Hand size:	2 cards
Community cards:	0
Betting rounds:	2

Hurricane might be thought of as 2-card stud poker. Hands consist of only 2 cards, and straights and flushes are not considered, so the only hand types possible are pairs and high-card hands. The first card to each player is dealt face up, followed by a round of betting. The second round of cards is dealt face down,and betting proceeds through a second round to the showdown [151]. The game may be enlivened by the addition of either 1 or 2 wild jokers; in the 2-joker case, a pair of jokers is considered the highest possible hand.

Hands of 2 cards are easy to count. In a deck without jokers, there are

$$13 \cdot \binom{4}{2} = 78$$

pairs and

$$\binom{13}{2} \cdot 4^2 = 1248$$

high-card hands. Taken together, these numbers add up to $\binom{52}{2} = 1326$, the number of ways to select 2 cards from a standard deck.

As in stud poker, high-card hands are ranked by the higher of their two cards, and this creates the same kind of inversion of rare hands that we examined on page 158, with the lowest-ranked hand, a 3-high hand consisting of a 3 and a 2, considerably rarer than the ace-high hand that sits atop the list of hands without a pair. The number of X-high hands, where X denotes the rank of the high card and jacks, queens, kings, and aces have $X = 11, 12, 13, 14$ respectively, is

$$4 \cdot 4(X - 2) = 16X - 32,$$

a value ranging from 16 for 3-high hands to 192 ace-high hands.

Adding one joker changes hurricane a bit. A hurricane hand holding a joker will always play as a pair; there are 52 new hands, each consisting of the joker with another card in the standard deck. This gives 130 pairs and the same 1248 high-card hands seen in the joker-free game.

With 2 jokers, there are $\binom{54}{2} = 1431$ possible hands, distributed as shown in Table 6.1.

TABLE 6.1: Hand counts for Hurricane with 2 wild jokers.

Hand	**Count**
Pair of jokers	1
Natural pair (no jokers)	78
Jokered pair	104
High card	1248

Table 6.1 shows that natural pairs are less common than pairs with a joker and so might be regarded as outranking them, so $W\ Q\heartsuit$ would lose to $Q\spadesuit\ Q\diamondsuit$. However, local custom might choose to regard all pairs as equal without concern for the number of jokers. Such a rule might carve out an exception for the only pair of jokers in the deck, assigning it the highest value—though this creates an unbeatable hand.

Using 1 or 2 jokers as bugs in hurricane effectively adds an ace or two to the deck, since there are no straights or flushes that a bug might otherwise complete. In a 2-bug game of hurricane, the probability of a pair of any rank other than aces is

$$\frac{\binom{4}{2}}{\binom{54}{2}} = \frac{6}{1431}.$$

This is less than half the probability of

$$\frac{6+8}{1431} = \frac{14}{1431}$$

of drawing a pair of a given rank, other than aces, with 2 fully wild jokers.

If a pair of jokers remains separated out as the top hand, then the probability of a pair of aces does not change when the jokers are downgraded to bugs.

Mambo Stud Poker

Deck composition:	52 cards
Hand size:	4 cards
Community cards:	1
Betting rounds:	3

Mambo stud poker, also known as mambo stud high-low split six-or-better in Massachusetts, is a variant that appeared in 1998. The Trump Taj Mahal Casino in Atlantic City, New Jersey briefly offered the game in its poker room. Players compete to make the highest and lowest 3-card poker hand from 4 cards including one community card.

As with 5-card stud, play begins with each player receiving two cards, one face down and one face up. Following a round of betting, each player is dealt one more face-up card. After the second round of betting, a face up community card is dealt to the table. The final round of betting takes place, and the pot is split between the highest and lowest hands, provided that the low hand is a 6-high or lower. In the absence of a sufficiently low hand, the high hand takes the entire pot.

High hands in mambo stud poker are ranked in the order given in Table 6.2.

TABLE 6.2: Mambo Stud Poker: High hand rankings.

Hand
High Mambo (suited AKQ)
Straight flush
Three of a kind
Straight
Flush
Pair
High card

Since straights and flushes do not count when determining the low hand, it is possible for a player to scoop the pot, either though a hand such as 543, which is a qualifying low hand and can also be played as a straight for high, or by choosing different cards for the high and low hands. A player holding $A♣\ K♣\ 6♣\ 2♡$ can play a club flush as his high hand and $62A$ for low.

The lowest possible hand in mambo stud poker is the "Low Mambo": $32A$ of any suits. Other low hands are ranked by their highest card: 4-high, 5-high, and 6-high. How many valid low hands are there?

A winning low hand requires one of the following $\binom{6}{3} = 20$ three-card combinations:

- 654
- 653
- 652
- 65A
- 643
- 642
- 64A
- 632
- 63A
- 62A
- 543
- 542
- 54A
- 532
- 53A
- 52A
- 432
- 43A
- 42A
- 32A

Since the low hand is made with 3 cards from a hand of 4, counting hands requires careful consideration of the fourth card. For example, the hand 6532 will play as 532 for low and should not be counted among the hands played as 653. It follows that any 3-card combination listed above can be combined with any fourth card ranking higher than its highest card. 6532 would thus be counted as a variant on 532, not 653.

If a 3-card low holding has highest card x, there are $4 \cdot (13 - x)$ higher cards that will complete it to a 4-card mambo stud poker hand. There are $\binom{x-1}{2}$ ways to choose the ranks of the other 2 cards, and $4^3 = 64$ ways to choose the suits of the 3 low cards. It is also possible to fill out a low hand to 4 cards by pairing one of the 3 low cards; this may be done in $3\binom{4}{2} \cdot 4^2 = 288$ ways regardless of the value of x.

Summing gives us

$$\sum_{x=3}^{6} \left[\binom{x-1}{2} \cdot (13 - x) \cdot 4^4 + 288\right] = 40,832$$

valid low hands, just over 15% of all 4-card hands.

At the other end of Table 6.2, there are $4 \cdot 48 = 192$ High Mambos.

Mexican Poker

Deck composition:	41 cards
Hand size:	5 cards
Community cards:	0
Betting rounds:	4

Mexican Poker is a variation on 5-card stud poker played with a 41-card deck. This deck removes all 8s, 9s, and 10s from the standard deck and adds a single joker, which is either fully wild or functions as a bug, depending on when it appears. In Mexican Poker, the jack and 7 are consecutive ranks, so $QJ765$ is a straight.

A hand of Mexican Poker begins with an ante from all players. Two cards, one face up and the other face down, are dealt to each player. After a round of betting, each player chooses whether or not to turn their hole card up, akin to Hold It and Roll It. A player who faces his hole card receives his third card face down, while a player who chooses not to turn up his hole card receives his third card face up. After the third cards are dealt, then, every player has 2 face-up cards and one face-down card. This is followed by a second round of betting, and the same process is repeated prior to the dealing of cards 4 and 5. In the showdown, each player has 4 face-up cards and 1 face-down card.

The joker is fully wild if it is dealt face down, even if the player later turns it face up. If a joker is dealt face up, it can only be used as a bug. With a wild joker in the deck, the highest possible hand is 5 of a kind. As in Section 1.6, we shall use a W to denote a wild joker, whether fully wild or a bug.

Using the formulas described beginning on page 177, we see that flushes outrank full houses in Mexican Poker, where the deck has 10 ranks and 4 suits: there are 980 flushes and 2160 full houses. This analysis does not account for the joker, though. Since the rules for the joker in this game are complicated, we will address the question of whether a single joker can raise the number of flushes beyond the number of natural full houses. Regardless of whether it's dealt face up or face down, a joker may always be used to complete a flush; if the number of flushes with a wild card does not bring the total number of flushes past 2160, then there is no need to investigate full houses with the joker to rank the hands properly.

The number of flushes with a joker is $4 \cdot \binom{10}{4} = 840$, which resolves the question at once: even without subtracting out the number of straight flushes with a joker, the number of flushes is less than the number of natural full houses. We shall count those straight flushes for completeness.

Since there are no 10s in the deck, the highest-ranked straight flush is $AKQJ7$, which is a royal flush in Mexican Poker. There are 28 natural straight flushes including royal flushes, since any card from ace to 7 can be the low card in a straight flush. For all straight flushes except $7JQKA$, 4 of the 5 cards—all except the lowest card—can be replaced by a wild joker and yield the same hand.

Example 6.5. A hand such as $W\ \heartsuit 3456$, where a deuce has been replaced by the joker, will be played as the higher straight flush $\heartsuit 34567$, where we value the joker as the missing $7\heartsuit$. ■

Since Mexican Poker rules always allow a joker to fill out a straight flush, there is no need to distinguish between face-up and face-down jokers. This gives $24 \cdot 4 = 96$ jokered straight flushes. For the straight flush starting with a 7, the hand $WJQKA$ would count the joker as a 7 in order to generate a straight flush. There are $4 \cdot 5 = 20$ ways to replace a card in a natural 7–ace straight flush by a joker. Adding gives a total of 116 jokered straight flushes, so the total number of flushes in Mexican Poker is $980 + 840 - 116 = 1704$, less than the 2160 natural full houses.

Three-of-a-kinds in Mexican Poker pose a different counting challenge. Natural three-of-a-kinds, without a joker, number

$$10 \cdot \binom{4}{3} \cdot \binom{9}{2} \cdot 4^2 = 23,040.$$

If a hand is raised to three of a kind by a wild joker, it must be of the form $Waabc$, where a, b, and c are cards of different ranks. There are

$$10 \cdot \binom{4}{2} \cdot \binom{9}{2} \cdot 4^2 = 34,560$$

hands of this form. Either the hand must contain a pair of aces or the joker must be dealt face down. Whether or not the joker is dealt face down depends on the player's strategy. If your hand includes a joker, what is the probability that it was dealt face down?

- If you always turn your hole card face up, then the probability is $\frac{4}{5}$. This is the maximum value for this probability.
- If you never turn your hole card up, the probability is $\frac{1}{5}$—the minimum possible probability.

The different treatment of face-up and face-down jokers is a strong incentive for players to expose hole cards if the joker has not yet appeared in another player's hand.

If the joker is dealt face up, there are 3 possibilities for the hand's final value. In the following list, the cards x, y, and z can be any rank except an ace.

1. The hand $WAAyz$ will play as three of a kind whether the joker is dealt face up or face down, because it will be used as an ace either way. It can be dealt in
$$\binom{4}{2} \cdot \binom{9}{2} \cdot 4^2 = 3456$$
ways.
2. A hand of the form $WAxxz$ with a face up joker scores as two pairs: aces and xs. There are
$$9 \cdot \binom{4}{2} \cdot 4 \cdot 8 \cdot 4 = 6912$$
ways to draw this hand.
3. Finally, if the hand is $Wxxyz$, which occurs in
$$9 \cdot \binom{4}{2} \cdot \binom{8}{2} \cdot 4^2 = 24,192$$
ways, and the joker is face up, the hand is only one pair.

A third possibility is a hand containing a natural three of a kind and a joker that cannot raise it to a full house or four of a kind; this would be a hand like $Wxxxy$, where neither x nor y denotes an ace. This presumes a face up joker that must be called an ace. There are

$$9 \cdot \binom{4}{3} \cdot 32 = 1152$$

hands of this type.

Trips

Deck composition:	54 cards
Hand size:	5 cards
Community cards:	0
Betting rounds:	4

At the Crystal Park Casino in Compton, California (now the Crystal Casino), *Trips* was a 5-card stud variant designed for an electronic gaming table, requiring a live dealer only for collecting fees and managing the physical chips wagered by players. Trips was played with a 54-card deck including 2 fully wild jokers, which made it easier for players to meet the game's requirement that the winning hand be at least 3 of a kind.

In Trips, all cards are dealt face up, with an ante round and betting rounds after the second, third, and fourth cards are dealt to each player. The highest qualifying hand wins the pot. If no hand qualifies with 3 of a kind or better, 20% of the pot is awarded to the player with the highest hand and the remaining 80% is left behind to start a new round. Players who have paid the collection fee of 50¢ to $5 for the first hand may continue play without an additional fee for the second hand of the round. Despite what we established in Section 1.6 about hand rankings with 2 wild cards in the deck, the rules for Trips follow the standard 5-card hand rankings, so 3 of a kind beats 2 pairs and 4 of a kind beats a full house.

Example 6.6. A hand with both jokers automatically qualifies with at least 3 of a kind. What is the probability of catching both jokers?

There is only 1 way to draw both jokers, and $\binom{52}{3}$ ways to fill out the 5-card hand. The probability of both jokers appearing in the same hand is

$$\frac{\binom{52}{3}}{\binom{54}{5}} = \frac{52 \cdot 51 \cdot 50 \cdot 5 \cdot 4}{54 \cdot 53 \cdot 52 \cdot 51 \cdot 50} = \frac{20}{54 \cdot 53} \approx .0070,$$

less than 1%. ■

How likely is it that an individual hand will fail to qualify? A non-qualifying hand must be two pairs or less and hold either 1 or 0 jokers. Since a hand with a natural pair and a joker will advance to 3 of a kind and qualify, a jokered non-qualifying hand must contain 4 cards that are neither sequential nor suited: $Wabcd$. The ranks may be chosen in $\binom{13}{4}$ ways, from which we subtract 10 to excise the hands containing a 4-card straight.That done, the suits may be selected in $4^4 - 4$ ways: each card may be 1 of 4 suits, and we subtract the 4 choices of suits that result in a 4-card flush. There are therefore

$$\left[\binom{13}{4} - 10\right] \cdot (4^4 - 4) = 177,660$$

such hands.

Non-qualifying hands without a joker are tabulated in Table 1.2 and number 2,524,332. Adding gives 2,701,992 non-qualifying hands, so the probability that a hand fails to qualify is

$$\frac{2,701,992}{\binom{54}{5}} \approx .8544,$$

so about 85% of 5-card hands will not qualify. At a 6-player table, a rough approximation of the probability that no one qualifies can be found by assuming that the hands are independent. We have

$$P(\text{No qualifying hands with 6 players}) \approx .8544^6 \approx .3890.$$

Slightly more than one 6-player hand in 3 will see no players qualify, and will move to a second hand.

WhoopAss Poker

Deck composition:	52 cards
Hand size:	8–9 cards
Community cards:	6
Betting rounds:	4

One of the worst names ever chosen for a game of chance was *WhoopAss Poker* [93]. Head-to-head WhoopAss Poker bears some similarities to Texas hold'em and to Omaha: players are dealt 2 hole cards and then 6 community cards are revealed, in three sets of 2 separated by betting rounds [74]. Between the second and third pair of community cards, players are given the option to buy another card, the "WhoopAss Card", and to designate it as a hole card, face down, or a community card for their exclusive use, dealt face up. Finally, the last pair of community cards, making a total of 6, is dealt face up. Each player's final hand must use exactly 2 of their 2–3 hole cards and 3 of their 6–7 community cards—this is where the designation of the WhoopAss Card as a hole card or community card matters.

The fee to buy the WhoopAss card in a limit game is set at the highest possible bet; in a 5/10 game, the card costs \$10. In a no-limit game, the cost of the WhoopAss card is the amount of the last previous bet, though a player who is all-in can buy the extra card for free. The house rakes \$1 from each fee paid; the remainder of the fees is deposited into the pot.

Example 6.7. With as many as 9 cards to choose from, WhoopAss Poker hands can run high. What is the probability of a 9-card hand without even a pair?

The probability of drawing 9 cards, including the WhoopAss card, of different ranks is simply

$$\frac{\binom{13}{9} \cdot 4^9}{\binom{52}{9}} \approx .0509,$$

or about once in 19.6 hands. Even this low probability is an overestimate, for it includes many straights and flushes. Reducing this probability to the correct value involves considering myriad subcases encompassing various intersections of straights and flushes among 9-rank hands.

Bill "Durango Bill" Butler opted for a more direct approach. Using a computer program to generate and evaluate all possible 9-card hands for their best 5-card subhand, Butler arrived at Table 6.3 for the frequencies of straights and flushes in 9-card hands.

TABLE 6.3: WhoopAss Poker: Frequency of 5-card straights and flushes among 9-card hands [13].

Hand	**Frequency**
Royal flush	713,460
Straight flush	5,874,656
Flush	453,008,864
Straight	509,071,920

Of course, these numbers include hands such as a 5-card straight flush with 4 pairs among the 9 cards that were not part of our original count. Butler's continued computations showed 69,788,040 9-card high-card hands, making the probability of a high-card hand a mere .0190 [13]. ■

WhoopAss Poker was patented in both head-to-head and carnival game formats, but was not all that successful either way. There is no record of how much this name functioned as a deterrent to players and casino officials in choosing games to play or to offer.

6.4 Exercises

Answers to starred exercises begin on page 338.

6.1.* Suppose that you are dealt a badugi. Find the probability that the player to your left also holds a badugi.

6.2.* What is the chance of a Three Flushes hand at Chinese Poker where each of the 3 hands is of a different suit?

6.3.* When using 2 jokers, the median hurricane hand is a high-card hand. What rank is that high card?

6.4.* Find the number of mambo stud poker hands which contain a qualifying low hand and also a pair of aces.

6.5.* Find the probability of 5 of a kind in Trips.

6.6.* Suppose that in a hand of WhoopAss Poker, you have 8 cards of different ranks among your hole cards and the 6 common community cards. If you choose to buy the WhoopAss card, what is the probability that its rank is different from the ranks of the 6 cards you can see?

(Why would you buy the 9th card? Perhaps your cards include several 4-card straights or flushes that might complete with one more card, and since the 6 community cards are of different ranks, there is little chance of an opponent completing a full house that beats both of those hands.)

6.7.* We know that it's impossible to complete a straight in 5 cards without holding either a 5 or a 10. How many 9-card WhoopAss Poker hands of 9 different ranks contain neither a 5 nor a 10, thus being candidates for a 5-card high-card hand?

6.8.* The Ocean's Eleven Casino in Oceanside, California offers a hybrid game called *Jacks Back*. The game begins as a standard 5-card draw poker game with jacks or better required to open. If no player holds a high enough opening hand, then the player to the left of the designated dealer position is required to open and the game proceeds as a lowball game. Assuming that the starting hands in a 6-player game are independent, find the probability that no one will be able to open with jacks or better.

Chapter 7

Poker-Based Carnival Games

A *carnival game* in a casino is any table game other than baccarat, blackjack, craps, or roulette. Many carnival games have been developed that include elements of poker. What sets these games apart from the games described in Chapter 6 is that players compete against the house and a fixed pay table rather than each other. While the dealer frequently draws a hand along with the gamblers, how that hand is played out is typically governed by house rules, so there is no room for dealers to vary their actions based on the players' hands.

Figure 7.1 shows a portion of the table felt from Caribbean Stud Poker, one of the first, and certainly one of the most successful, table games with a poker flavor. While its early history is murky, it is certain that the game was first devised in the 1980s prior to the Texas hold'em boom. Along the way, the inventors sold the game to Mikohn Gaming for $30 million, inspiring many game designers as they sought to craft an equally lucrative game [81]. See page 223 for a discussion of the game's mathematics.

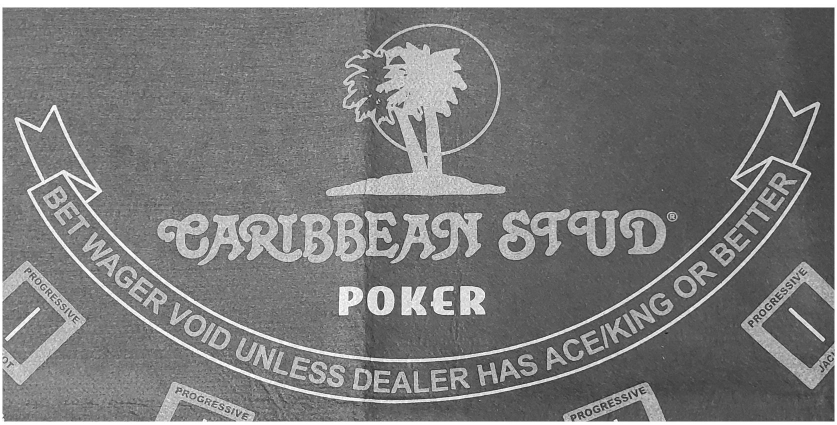

FIGURE 7.1: Caribbean Stud Poker table felt.

Many of these games have common betting structures, beginning with a required Ante bet before any cards are dealt. Other wagers, both mandatory and optional, may be available at the same time before the deal. After some of the cards are dealt, players may have the opportunity to fold, losing their Ante bet, or to make an additional Play bet which buys the right to see additional cards or to enter the final showdown. The amount of the Play bet is typically

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keyed to the size of the ante and limited to some fixed multiple of it. There may be optional side bets which are won when a player's hand is sufficiently high.

7.1 Hold'em Derivatives

Part of the American poker boom in the early 21st century was a rush to modify poker into a more standard table game, with players playing against a dealer or a fixed pay table rather than each other. As Texas hold'em was the most popular poker variant at the time, it was natural that some of these new games were variations on that. Games in this section share with Texas hold'em the feature that players combine personal cards with community cards to make their best 5-card hand. The number of community cards may vary from 5: as low as 3 or as high as 6.

Ultimate Texas Hold'em

Deck composition:	52 cards
Hand size:	7 cards
Community cards:	5
Betting rounds:	2–4
Required bets:	Ante, Blind
Optional bets:	Play, Trips

Ultimate Texas Hold'em (UTH) was one of the most successful table games based on Texas hold'em. UTH began as a multiplayer video poker game before migrating to the gaming floor as a live game dealt with actual cards.

The player's goal in UTH is to build a higher 5-card hand than the dealer's from two private hole cards and 5 community cards. Players do not compete against each other for highest hand; they only play individually against the dealer. Fundamental to UTH, and to some other carnival games with roots in poker, is the notion of a dealer *qualifying hand*. This represents a minimum threshold that the dealer's completed hand must meet before players can win against it. In UTH, the dealer qualifies with a final hand of at least a pair; if the dealer fails to qualify, all Ante bets push, but other bets sill have action.

A hand of UTH begins with each player making two equal bets: the Ante and Blind bets. An optional Trips bet may also be made at this point. Players and dealer are then each dealt 2 hole cards. Betting from this point forward favors the early bettor: the amount that a gambler may wager once cards are dealt decreases as the hand elapses. After the hole cards are dealt, players may either check or bet 3 or 4 times the amount of their Ante bet. This bet is called the Play bet, and may only be made once per hand.

The dealer then delivers the three-card flop. Players who have not made a Play bet may either check or make a Play bet of double their Ante bet.

Example 7.1. Optimal UTH strategy (to be described later) calls for an immediate raise with any ace in the hole. If the player's hole cards are $A\heartsuit\ 2\diamondsuit$, she should make a 4× raise; the option of raising 3 times the Ante should never be used, as a hand worth raising is worth raising as much as allowed. The 3× raise option simply affords players an opportunity to give the house a larger advantage by playing less than optimally.

If the flop is then $J\heartsuit\ 9\heartsuit\ J\spadesuit$, everyone, including the dealer, holds at least a pair of jacks, and so the dealer is certain to qualify. Since our player already made a 4× Play bet, she may make no further raises, and awaits the showdown. ■

The final two community cards are then exposed together rather than as separate turn and river cards, and players who have yet to make a Play bet must either match their Ante with a 1× Play bet or fold. At this point, a player who has not yet folded is either wagering 4, 3, 2, or 1 times their original Ante bet in addition to the Ante itself.

With the betting complete, hands are now exposed. If the dealer qualifies but loses to the player's hand, the Ante and Play bets are paid even money. The Blind bet pays according to Table 7.1 if the player's hand beats the dealer's and is at least a straight; it pushes if the player beats the dealer with a hand lower than a straight.

TABLE 7.1: Ultimate Texas Hold'em: Blind bet pay table. Player's hand must beat the dealer.

Hand	**Payoff odds**
Royal flush	500–1
Straight flush	50–1
4 of a kind	10–1
Full house	3–1
Flush	3–2
Straight	1–1

Example 7.2. Continuing the hand above, suppose that the final two cards are $K\heartsuit\ Q\diamondsuit$. The player's final hand is $J\heartsuit\ J\spadesuit\ A\heartsuit\ K\heartsuit\ Q\diamondsuit$. If the dealer's hole cards are $2\spadesuit\ 7\clubsuit$, then the dealer's best hand consists of the 5 community cards: $J\heartsuit\ J\spadesuit\ K\heartsuit\ Q\diamondsuit\ 9\heartsuit$. The player hand of a pair of jacks with an ace kicker wins on her Ante and 4× Play bets, while her Blind bet pushes. ■

Texas hold'em derivatives frequently play out very similar to the base game, so many of the most interesting questions arise when looking at side bets. The optional Trips bet pays off if a player's hand is at least 3 of a kind, with no requirement that the hand beat the dealer—but also no saving push if a low hand beats the dealer. Trips pays off as in Table 7.2. Our player from

TABLE 7.2: Ultimate Texas Hold'em: Trips bet pay table.

Hand	**Payoff odds**
Royal flush	50–1
Straight flush	40–1
4 of a kind	30–1
Full house	8–1
Flush	6–1
Straight	5–1
3 of a kind	3–1

Examples 7.1 and 7.2, holding only a pair, would lose this bet.

Example 7.3. The Trips bet wins for any player making it if the 5 community cards contain at least 3 of a kind, which is a possibility that can add to the game's excitement. What is the probability of this event?

While different players may receive different payouts on a Trips bet won on the board, depending on how their hole cards might improve the 5 community cards, this probability may be computed by looking at the community cards as a 5-card hand. The various winning hands have the frequencies shown in Table 7.3, and so the number of ways to make a winning board is found by

TABLE 7.3: Ultimate Texas Hold'em: Board hand frequencies.

Hand	**Frequency**
Royal flush	4
Straight flush	36
4 of a kind	624
Full house	3744
Flush	5108
Straight	10,200
3 of a kind	54,912

summing the second column. We have

$$P(\text{Board wins Trips}) = \frac{74,628}{2,598,960} \approx .0287 \approx \frac{1}{35}.$$

■

To compute the expected value of a \$1 Trips bet, we need to consider all 7 cards from which a player chooses 5 to set his hand.

Example 7.4. There are four 5-card royal flushes, any of which may be combined with any 2 of the remaining 47 cards to create a 7-card hand containing a royal flush. The total number of 7-card royal flush hands is then

$$4 \cdot \binom{47}{2} = 4 \cdot 1081 = 4324.$$

Using this approach to count the number of ways to achieve other poker hands in 7 cards requires some attention to the possibility of moving the hand up to a higher-ranking hand. For example, combining the full house

$$3\heartsuit\ 3\clubsuit\ 3\diamondsuit\ 2\diamondsuit\ 2\spadesuit$$

with 2 other cards must exclude any combination including the 3♠ or the 2♡ and 2♣, both of which create a 7-card four-of-a-kind hand. There are 47 such pairs. Additionally, the two extra cards cannot be a pair of a rank higher than 2, for that would generate a full house outranking 33322. This excludes $11 \cdot \binom{4}{2} = 66$ further pairs, with the number diminishing as the rank of the pair in the initial full house increases.

This applies to full houses where the triple of cards outranks the pair. If the pair ranks higher than the triple, as, for example, with

$$6\heartsuit\ 6\clubsuit\ 6\diamondsuit\ J\diamondsuit\ J\spadesuit,$$

then it is also necessary to exclude pairs of extra cards that add a third card of the pair's rank, as these would move the given full house to a higher-ranking full house.

Pulling everything together gives a total of 3,473,184 7-card hands containing a full house that is counted as a full house. ■

There are $\binom{52}{7} = 133,784,560$ ways to choose 7 cards from a standard deck; the frequencies of the standard poker hands are shown in Table 7.4. We note that the relative ranking of hands changes with 7 cards to choose from: pairs and two-pair hands are both more common than no-pair hands.

The resulting expectation of a \$1 Trips bet, with Table 7.2 as pay table, is −\$.0190, giving the casino a low edge of 1.90%.

The Wizard Strategy

In developing a strategy for raising UTH hands, the challenge is to raise winning hands as early as possible so as to maximize the total amount won. At the same time, the gambler seeks to avoid throwing good money after bad hands. Finding the best balance between these two goals is a task for simulation.

TABLE 7.4: 5-card poker hand frequencies with 7-card hands.

Hand	**Frequency**
Royal flush	4324
Straight flush	37,260
Four of a kind	224,848
Full house	3,473,184
Flush	4,047,644
Straight	6,180,020
Three of a kind	6,461,620
Two pairs	31,433,400
One pair	58,627,800
No pair	23,294,460

The following strategy is called the "Wizard Strategy", and is explained online by Michael Shackleford, the "Wizard of Odds" [135]. At the 4× or 3× stage, with the understanding that any hand worth raising at this point is worth backing with a 4× raise, the following hole cards should be raised 4×:

- **Any pair of 3s or greater.** A pair of 2s is not, at this point, good enough to back with a maximum raise. If the dealer should qualify, a pair of 2s by itself can only hope to tie the dealer, and that is a longshot.
- **Any hand containing an ace.** An ace in the hole is an ace that the dealer cannot hold, and so represents a considerable advantage.
- **Any hand containing a king with a 5 or higher, or a king with a suited 2–4.** Suited hands with lower cards are included because of the possibility of a flush draw among the community cards.
- **Any queen with an 8 or higher, or a queen with a suited 6 or 7.**
- **Any JT, suited J9, or suited J8.**

Of the $\binom{52}{2} = 1326$ different possible two-card hands, 500 of them are worth a 4× raise. For the remaining hands, the community cards help to make the call on the later, smaller raises. The 2× raise, which may be made after seeing 5 cards, is advisable under the following circumstances:

- **Any hand holding two pairs or better.**
- **Any hidden pair except pocket deuces.** A *hidden* pair is a pair using one of the player's hole cards. Hidden pairs are so named because they are not seen by the dealer—a pair of 7s in the community cards is not a hidden pair, and it can be part of both dealer and player hands. A pair of 2s—the lowest possible pair—is still not strong enough to merit a 2× raise.

- **Four suited cards including a hidden 10, face card, or ace.** These hands are raised due to their potential to draw into a flush or a high pair that is not matched by the dealer since the high card is hidden.

Example 7.5. If your hole cards are $T\diamondsuit$ $3\clubsuit$ and the flop is $9\diamondsuit$ $8\diamondsuit$ $2\diamondsuit$, then the four-card flush merits a $2\times$ raise. Replacing the $T\diamondsuit$ by the $4\diamondsuit$ turns the hand into a 4-card flush that does not justify a $2\times$ raise, because the flush no longer contains a high card. ■

Finally, when all 7 cards are exposed, the player should raise any hand with a hidden pair or better, now including a pair of 2s, or one with fewer than 21 dealer outs. An *out* in UTH is a card that, if held by the dealer, can be combined with the community cards to form a 5-card hand that beats the player.

Example 7.6. Suppose that you hold $9\diamondsuit$ $8\diamondsuit$ and the flop is $T\spadesuit$ $6\clubsuit$ $2\diamondsuit$, so you have not yet raised. If the turn and river are $Q\clubsuit$ $6\spadesuit$, then your best hand is $6\spadesuit$ $6\clubsuit$ $Q\clubsuit$ $T\spadesuit$ $9\diamondsuit$: a pair of 6s with a queen kicker. Your hand can be beaten if the dealer holds any of the following cards:

- Any ace or king, which will give the dealer a higher kicker to his pair of 6s. (8 total outs)
- Any jack, which allows the dealer to beat your 10 as second kicker with $66QJx$. (4 outs)
- Any queen, 10 or 2, which give the dealer two pairs. (9 outs)
- Either of the two remaining 6s, which give the dealer 3 of a kind. (2 outs)

Since this list includes 23 outs, you should fold rather than raising $1\times$. ■

At the point where the final raise-or-fold decision is to be made, there are 45 unknown cards remaining in the deck. If the dealer has exactly 21 outs, the probability that his hole cards contain at least one of them is

$$1 - \frac{\binom{24}{2}}{\binom{45}{2}} = \frac{119}{165} \approx .7212.$$

More dealer outs increases this probability. For simplicity, we ignore the small probability of two hole cards improving the dealer's hand though neither can do so alone, such as a pair of 5s in the hand above. Your hand will lose to the dealer's nearly ¾ of the time, and so is not worth additional wagering.

Action Hold'em

Deck composition:	52 cards
Hand size:	7 cards
Community cards:	5
Betting rounds:	3
Required bets:	Ante
Optional bets:	Flop, Turn & River, 7–2, Any Pair

Action hold'em is a variant played in the United Kingdom that mimics the structure of Texas hold'em while fixing the bet amounts [6]. Players begin by placing an Ante bet, after which each player and the dealer is dealt 2 cards face up—since all players face off individually against the dealer, there is no need to conceal cards. Seeing other players' cards may aid in making betting decisions. Players may then either fold or make a Flop bet of twice their ante, which buys the right to see the first 3 community cards. After these cards are dealt, players may then make a Turn & River bet of three times their ante or may check and stay in action risking only the Ante and Flop bets. The final two community cards are then dealt to the table, and each player's best 5-card hand is compared to the dealer's best hand. Winning hands are paid according to a sliding scale which takes the relative strength of both hands into account [6].

Action hold'em also includes two optional prop bets: *7–2* and *Any Pair*, which players may make on their own hands or on the dealer's hand. The 7–2 bet pays off if the selected hand is dealt a 7 and a 2 as its first two cards; as we saw on page 112, this is the worst possible starting hand if unsuited. 7–2 pays off even if the 7 and 2 are of the same suit; this makes the winning probability

$$\frac{4 \cdot 4}{\binom{52}{2}} = \frac{8}{663} \approx .0121.$$

Since the bet pays off at 72–1 when it wins, the expectation is

$$E = (72) \cdot \frac{8}{663} + (-1) \cdot \frac{655}{663} = -\frac{79}{663} \approx -\$.1192,$$

so the casino holds an 11.92% advantage.

The Any Pair bet pays off at 12–1 if the selected hand is dealt a pair up front. The probability of winning this bet, regardless of the hand, is

$$\frac{13 \cdot \binom{4}{2}}{\binom{52}{2}} = \frac{78}{1326} = \frac{1}{17}.$$

With a 12–1 payoff and odds against of 16–1, the casino enjoys a healthy edge on this bet: 23.53%.

All in Hold'em

Deck composition:	52 cards
Hand size:	7 cards
Community cards:	5
Betting rounds:	2
Required bets:	Ante
Optional bets:	Player Hole Cards, Final Hand

All in hold'em adds an element of bluffing to a carnival poker game. In one version of the game, each player antes and is then dealt 2 cards. The dealer also receives 2 face-down hole cards. The player's choices at this point affect the subsequent gameplay.

- The player may *fold*, ending their participation in the hand.
- The player may *raise* by making a second bet of 5× the ante.
- The player may *go all-in* by making a second bet of 10× the ante.

The dealer will call each player's bet as follows:

- If the player has merely raised, the dealer will call if his hand is a pair or has a blackjack value of at least 13, with aces counting as 11. It follows that if the dealer has an ace in hand, he will call any player who raises.
- If the player has gone all-in, the dealer will call if his hand is a pair or has a blackjack value of at least 17.

A strategy choice comes into play in all-in hold'em when a player might choose to go all-in on a weak hand in the hope that the dealer will fail to meet the higher standard to call the bet. If some players raise and some go all-in, the dealer will play against each player according to their choice, so a dealer may fold against some players while calling against others. If the dealer folds, player antes are paid at even money and their subsequent bets push.

If the dealer calls one or more hands, 5 community cards are dealt, and each player compares his or her best 5-card hand to the dealer's. The dealer has no qualifying hand, and the higher hand wins. A winning player receives even money on their Ante bet and their raise.

Example 7.7. Three players make their ante bets and receive 2 cards apiece.

- Chris opts to go all-in on 7♣ 6♣.
- Pat is dealt $A\diamondsuit$ 4♢, and raises.
- Alex holds a weak hand, 6♡ 4♠, but goes all-in as a bluff.

The dealer's hand is $K\heartsuit\ 6\diamondsuit$. With a blackjack value of 16, the dealer calls Pat, but folds against Chris and Alex, who each receive even money on their antes. Their 10× all-in bets are returned. Alex's bluff was successful.

The community cards are then dealt for Pat and the dealer:

$$A\spadesuit\ J\spadesuit\ 5\clubsuit\ 2\spadesuit\ 2\clubsuit.$$

Pat holds 2 pairs: aces over deuces. The dealer holds a pair of deuces with an ace kicker, so Pat wins a total of 6 times the initial wager. ■

What are the different probabilities of the dealer calling a player's raise or all-in bet?

The probability of a dealer pair is

$$\frac{52}{52} \cdot \frac{3}{51} = \frac{1}{17} \approx .0588.$$

A dealer total of 13 can be reached with a 2 and ace, 3 and 10 (or face card), 4 and 9, 5 and 8, or 6 and 7. The probability of a 13 with 2 non-10s is

$$2 \cdot \frac{4}{52} \cdot \frac{4}{51} = \frac{4}{663},$$

where the factor of 2 accounts for the fact that the 2 cards may be dealt in either order. The probability of a 13 consisting of a 3 and a 10-count card is

$$2 \cdot \frac{4}{52} \cdot \frac{16}{51} = \frac{16}{663}.$$

Adding across all possibilities gives

$$P(\text{Dealer 13}) = \frac{20}{221}.$$

A dealer 14, other than a pair of 7s, can be dealt as a 3 and ace, 4 and 10, 5 and 9, or 6 and 8. Following the reasoning above gives

$$P(\text{Dealer 14, no pair}) = \frac{56}{663}.$$

Table 7.5 shows the probabilities of each total, without pairs, from 13–21. Adding everything up gives

$$P(\text{Dealer calls with pair or greater than 13}) = \frac{451}{663} \approx .6802,$$
$$P(\text{Dealer calls with pair or greater than 17}) = \frac{231}{663} \approx .3484.$$

The probability of a dealer calling with 17 or more, or a pair, is roughly half the chance of calling on a pair or a hand at least 13.

TABLE 7.5: All-in Hold'em: Probability of dealer 2-card total from 13–21, without a pair.

Total	**Probability**
13	.0905
14	.0845
15	.0845
16	.0724
17	.0724
18	.0603
19	.0603
20	.0483
21	.0483

All-in hold'em players may make an optional prop bet on their initial 2 cards. The Player Hole Cards bet pays off if the cards are suited or a pair, per Table 7.6.

TABLE 7.6: All-in hold'em: Player Hole Cards pay table.

Hole cards	**Payoff**
2 red aces	50–1
Suited AK	25–1
Pair of aces	20–1
Pair of face cards	8–1
Pair, 6s–10s	3–1
Pair, 2s–5s	2–1
Suited cards	1–1

Since there are only $\binom{52}{2} = 1326$ possible initial 2-card cards, evaluating the probabilities involved in the Player Hole Cards bet is simple. The probability of 2 suited cards, other than the suited AK that pays off at 25–1, is

$$\frac{4 \cdot \binom{13}{2} - 4}{\binom{52}{2}} = \frac{308}{1326} \approx .2323.$$

The HA of this bet is computed in Exercise 7.3. A second optional bet is available on the player's final 5-card hand. Final Hand pays off using Table 7.7.

TABLE 7.7: All-in hold'em: Final Hand pay table.

Hand	**Payoff**
Royal flush	500–1
Straight flush	100–1
4 of a kind	40–1
Full house	8–1
Flush	6–1
Straight	4–1
3 of a kind	2–1

Dakota Duel Draw

Deck composition:	52 cards
Hand size:	5 or 8 cards
Community cards:	2 sets of 3
Betting rounds:	2
Required bets:	Hand 2
Optional bets:	Big 8, Hand 1

Dakota Duel Draw is a game approved for play in the casinos of Deadwood, South Dakota. Each player is dealt 2 hole cards, which may be combined with either of 2 sets of 3 community cards to form a 5-card poker hand that is paid against a pay table. The dealer takes no cards in Dakota Duel Draw. A separate bet, the Big 8 bet, can be made on the best 5-card hand made from all 8 cards.

Prior to the deal, players bet on the Big 8 spot or on hand 2. Each player receives 2 hole cards, and 3 cards are dealt to hands 1 and 2. After viewing their hole cards, players may fold and forfeit a bet on hand 2 or make an additional bet on hand 1 equal to their hand 2 bet. The Big 8 bet, if made, remains in action regardless of the player's decision on his or her own hand. Bets on hands 1 and 2 pay off if the hand is at least a pair of 9s; the Big 8 bet requires a straight or higher.

Example 7.8. Three players all make Big 8 and hand 2 bets, and are dealt the following hole cards:

- Robin: $T♠$ $6♢$.
- Chris: $A♠$ $5♠$.
- Pat: $J♢$ $5♢$.

All 3 players stay in and make a bet on hand 1. The community cards are then revealed:

- Hand 1: $J♣$ $T♡$ $8♠$.

- Hand 2: $Q\clubsuit$ $7\diamondsuit$ $3\diamondsuit$.

Pat (pair of jacks) and Robin (pair of 10s) both hold winning pairs in Hand 1. No one holds a winning Hand 2. ■

Hands 1 and 2 pay off according to Table 7.8.

TABLE 7.8: Dakota Duel Draw: Pay table for hands 1 and 2.

Hand	Payoff
Royal flush	100–1
Straight flush	50–1
4 of a kind	30–1
Full house	8–1
Flush	6–1
Straight	4–1
3 of a kind	3–1
2 pairs	2–1
Pair, 9s or higher	2–1

A Big 8 pay table is shown here as Table 7.9.

TABLE 7.9: Dakota Duel Draw: Pay table for Big 8 wager.

Hand	Payoff
Royal flush	50–1
Straight flush	30–1
4 of a kind	15–1
Full house	5–1
Flush	3–1
Straight	2–1

Assessing the Big 8 wager requires consideration of 8-card hands and how many of them contain a given 5-card poker hand. For royal flushes, all that is necessary is that the 5 cards of the flush appear among the 8 cards. This includes the possibility that the 5 cards of the royal flush all appear among the 6 community cards, which would lead to a big loss for the casino if players made the Big 8 bet.

It does not matter where on the board the cards appear; all we need is that 5 of the 8 cards form a royal flush. The other 3 cards can be anything in the deck, so the probability is

$$\frac{4 \cdot \binom{47}{3}}{\binom{52}{8}} = \frac{64,860}{752,538,150} \approx 8.6188 \times 10^{-5},$$

or once every 11,602.5 hands. The casino's risk here is very small.

For four-of-a-kind hands, computing the exact probability requires two steps: We begin by computing the probability of 4 of a kind among 8 cards, as we did above, but then must subtract the probability that the 8 cards contain two 4 of a kinds. The probability that an 8-card hand contains two 4-of-a-kinds is

$$\frac{\binom{13}{2}}{\binom{52}{8}} = \frac{1}{9,647,925},$$

since the only choice is the ranks present in the hand. We have

$$P(4 \text{ of a kind in 8 cards}) = \frac{13 \cdot \binom{48}{4}}{\binom{52}{8}} - \frac{\binom{13}{2}}{\binom{52}{8}} \approx .0034 \approx \frac{1}{298}.$$

Any player making a bet on hand 1 qualifies for a Pocket Pair bonus if their hole cards form a pair. One version of the Pocket Pair pay table pays 4–1 on a pair of 5s through aces and 3–1 on a pair of 2s, 3s, or 4s. A player dealt a pocket pair is thus nearly certain to make a bet on hand 1 instead of folding, since the smaller of the two payoffs matches 3 other bets and guarantees at least a break-even outcome. Since this bonus requires no additional player wager, its expected value is positive. The probability of a pocket pair of a given rank is $\frac{1}{221}$, which makes the value of the Pocket Pair bonus

$$(4) \cdot \frac{10}{221} + (3) \cdot \frac{3}{221} = \frac{49}{221} \approx \$.22$$

per dollar wagered on hand 1.

Fast Action Hold'em

Deck composition:	312 cards
Hand size:	9 cards
Community cards:	5
Betting rounds:	2
Required bets:	Ante
Optional bets:	Bonus

Fast Action Hold'em was introduced in 1991 at Harvey's Casino in Stateline, Nevada, and has since been offered at a number of casinos in the USA and Canada. Game designer Bee Estes, a Harvey's employee, set out to develop a game combining the skill set of poker with the simplicity and speed of blackjack [90].

The game is a hold'em variant dealt from a 6-deck shoe, which provides seven new hands that are broken out in the hand rankings. Among these hands are 5 of a kind and several hands incorporating suited cards; a suited 5 of a kind is the highest-ranking hand. Table 7.10 situates these new hands among the ten standard poker hands.

TABLE 7.10: Fast Action Hold'em hand rankings [90].

Suited 5 of a kind
Royal flush
Suited 4 of a kind
Straight flush
Flush with a full house
Five of a kind
Flush with 3 of a kind
Flush with 2 pairs
Flush with a pair
Four of a kind
Full house
Flush
Straight
Three of a kind
Two pairs
Pair
High card

Players make an initial Play bet and are dealt 4 hole cards; the dealer also takes 4 cards. Two must be discarded before the community cards are dealt. The dealer must follow a house way for discarding; the approved house way in the state of Washington is shown in Table 7.11. The dealer should read down Table 7.11 until she finds a combination that she holds, then discard down to it.

If there is a choice among competing discards at one level, as when holding two suited pairs, the house way calls for retaining the highest option. This house way is a good strategy for players as they seek to make the best choice from among their hole cards.

Example 7.9. Most of the time, a player dealt a pair in the hole will hold it; the only exception is when a low pair (2s through 7s) is dealt together with an ace and a face card. This has probability

$$p = \frac{6 \cdot \binom{24}{2} \cdot 24 \cdot 72}{\binom{312}{4}} \approx .0074.$$

TABLE 7.11: Fast Action Hold'em: House way in Washington state [59].

Cards to hold	**Example**
Pair of 8s or higher	$J\clubsuit\ J\spadesuit$
Ace with face card	$A\heartsuit\ J\diamondsuit$
Suited pair, 2s–7s	$4\diamondsuit\ 4\diamondsuit$
Unsuited pair, 2s–7s	$3\clubsuit\ 3\heartsuit$
Ace with suited card	$A\clubsuit\ 9\clubsuit$
Suited cards, both 10 or higher	$Q\spadesuit\ T\spadesuit$
Unsuited cards, both 10 or higher	$K\diamondsuit\ J\heartsuit$
Ace with unsuited card	$A\spadesuit\ 9\clubsuit$
Face card with suited card	$J\clubsuit\ 7\clubsuit$
Face card with unsuited card	$Q\heartsuit\ 6\diamondsuit$
Suited connectors	$7\spadesuit\ 6\spadesuit$
Unsuited connectors	$7\diamondsuit\ 6\clubsuit$
Suited cards, 2–10	$8\clubsuit\ 5\clubsuit$
Unsuited cards, 2–10	$T\heartsuit\ 8\diamondsuit$

■

The 5 community cards are all dealt to the table in one round, and players then try to beat the dealer's best 5-card hand. Players may use 0, 1, or 2 of their hole cards to build their bets 5-card hand. In the original version of the game, winning bets were paid at 19–20 odds, charging a 5% commission that is the source of the casinos' edge. More recently, winners are paid even money, and the house gets its edge by taking all ties.

In Massachusetts, a player dealt four of a kind need not play out the hand, but may face his or her cards immediately and claim a 5–1 payoff [97]. This hand is called a *natural*; its probability is

$$\frac{13\cdot\binom{24}{4}}{\binom{312}{4}} \approx 3.5669\times 10^{-4}.$$

If the dealer is dealt a natural, all players lose immediately.

In venues where the 5–1 payoff for a natural is not available, players may have the option to split a good initial 4-card hand into two 2-card hands by matching their Ante bet. This is analogous to splitting a pair in blackjack. Four of a kind would be a good candidate for splitting.

Example 7.10. What is the probability of a flush with 5 of a kind, the highest-ranked hand in Fast Action hold'em?

Let X denote the card, both rank and suit, that appears 5 times. Assuming a hand of 5 Xs dealt from the top of a 6-deck shoe, we must consider the distribution of the Xs between the player's hole cards and the community cards. A player dealt $XXXX$ cannot draw into 5 of a kind on the community cards; splitting the four cards or claiming a natural and its 5–1 payoff is probably a sound choice, for reasons exceeding the impossibility of a suited 5 of a kind.

If the player holds k Xs, where k runs from 0 to 3, the remaining Xs must appear on the board. We look to compute

$$\sum_{k=0}^{3} P(\text{Dealt } k\ X\text{s}) \cdot P(\text{Board contains } (5-k)X\text{s} \mid \text{Dealt } k\ X\text{s}).$$

Careful counting gives

$$P(\text{Dealt } k\ X\text{s}) = \frac{52 \cdot \binom{6}{k} \cdot \binom{306}{4-k}}{\binom{312}{4}}$$

and

$$P(\text{Board contains } (5-k)X\text{s} \mid \text{Dealt } k\ X\text{s}) = \frac{\binom{6-k}{5-k} \cdot \binom{312-(4-k)}{5-(5-k)}}{\binom{308}{5}}$$

$$= \frac{\binom{6-k}{5-k} \cdot \binom{308+k}{k}}{\binom{308}{5}}.$$

Adding everything up gives 1.6291×10^{-6} as the approximate probability of a suited 5 of a kind; about 1 chance in 613,849. ■

Example 7.11. Another new hand in Fast Action hold'em is a flush with a full house. How frequent are these?

In a flush with a full house, the full house accounts for 5 of the 7 cards. There are $52 \cdot \binom{6}{3} \cdot 12 \cdot \binom{6}{2} = 187,200$ ways to do this. Cards 6 and 7 must then be chosen so that they do not elevate the hand to a suited 4 of a kind or 5 of a kind. For example, if the full house is $8♠\ 8♠\ 8♠\ J♠\ J♠$, the final 2 cards can be anything except another $8♠$ with any other card or a pair of $J♠$s.

This hand occurs in

$$52 \cdot \binom{6}{3} \cdot 12 \cdot \binom{6}{2} \cdot \left[\binom{307}{2} - 3 \cdot 306 - \binom{4}{2}\right] = 8,619,998,400$$

ways. This is a large number, but it's small in comparison with the number of 7-card hands, which is

$$\binom{312}{7} = 53,359,916,132,952.$$

■

In Washington state, players may make an optional Bonus bet that wins if the player's hand is at least 3 of a kind, though 3 of a kind merely pushes. The Bonus wager is paid even if the player's hand loses to the dealer's. A pay table for the Bonus bet which carries a 4.63% house edge is shown in Table 7.12.

TABLE 7.12: Fast Action Hold'em: Bonus bet pay table [59].

Player's hand	**Payoff odds**
Suited 5 of a kind	1000–1
Royal flush	200–1
Straight flush	75–1
5 of a kind	40–1
4 of a kind	7–1
Full house	3–1
Flush	2–1
Straight	2–1
3 of a kind	Push

Players making a Bonus wager of at least $5 are eligible for an Envy payoff if another player at the same table holds at least 5 of a kind. The Envy Bonus pays fixed amounts rather than paying odds on a player's Bonus bet; its pay table is Table 7.13.

TABLE 7.13: Fast Action Hold'em: Envy Bonus payoffs [59].

Player's hand	**Envy Bonus**
Suited 5 of a kind	$1,000
Royal flush	$250
Straight flush	$50
5 of a kind	$10

If multiple players hold very high hands—which would happen if the community cards by themselves formed a sufficiently high hand—the Envy Bonus may be paid multiple times. Players do not qualify for the Envy Bonus based on their own hand or the dealer's hand. Assuming that the game is dealt from a full 312-card shoe, what is the probability that the community cards, by themselves, trigger the Envy Bonus?

The community cards must form 5 of a kind—suited or unsuited—or a royal or straight flush. For convenience, we compute $P(5 \text{ of a kind})$ without separating into suited and unsuited cases. We have

$$P(5 \text{ of a kind}) = \frac{13 \cdot \binom{24}{5}}{\binom{312}{5}} \approx 2.3162 \times 10^{-5}.$$

$$P(\text{Royal flush}) = \frac{4 \cdot 6^5}{\binom{312}{5}} \approx 1.3038 \times 10^{-6}.$$

$$P(\text{Straight flush}) = \frac{36 \cdot 6^5}{\binom{312}{5}} \approx 1.1734 \times 10^{-5}.$$

Adding these three probabilities gives 3.6200×10^{-5}, approximately 1 chance in 27,625.

7.2 Stud Poker Variations

Caribbean Stud Poker

Deck composition:	52 cards
Hand size:	5 cards
Community cards:	None
Betting rounds:	2
Required bets:	Ante
Optional bets:	Call

Caribbean stud poker (CSP) is a great success among carnival games. After nearly a decade in Caribbean resorts and cruise ships, the game came to Las Vegas in 1992, at Bally's (now the Horseshoe) and the Rio [48]. CSP players pit their 5-card hands individually against the dealer's. While multiple players may bet against a single dealer hand, players are not in competition with one another.

Play begins with each player making an initial Ante bet. Each player is dealt five cards from a single deck, and five cards are dealt to the dealer. No further cards will be dealt. One dealer card—the *upcard*—is turned up to give the players some information about their common opponent. Based on the strength of their hands and the dealer's upcard, players may either fold, forfeiting their ante, or make an additional Call bet of double their Ante bet that their hand will beat the dealer's. This decision is the lone place where skill becomes important; unlike in standard poker, everything else in CSP is a matter of chance.

Example 7.12. You have made a $1 ante bet and are dealt the following CSP hand:

$$A\clubsuit\ K\diamondsuit\ Q\diamondsuit\ 9\heartsuit\ 2\spadesuit.$$

The dealer's upcard is the 8♡. The question you now face is whether or not your AK-high hand is likely to beat the dealer's hand, which is starting with a card lower than 4 of yours. ■

At this point, the game is essentially even. The house derives its advantage from how the hands are played out. The dealer's hand qualifies if it is least ace-king high. If the dealer fails to qualify, the players who have not folded are paid even money only on their Ante bets, and the Call bets push. If the dealer qualifies, the hands are compared, and if the player's hand beats the dealer's, the Ante bet is paid at 1 to 1 and the Call bet is paid in accordance with Table 7.14 on page 224.

TABLE 7.14: Caribbean stud poker: Call bet payoffs in live games.

Player hand	**Payoff odds**
Royal flush	100–1
Straight flush	50–1
Four of a kind	20–1
Full house	7–1
Flush	5–1
Straight	4–1
Three of a kind	3–1
Two pairs	2–1
One pair	1–1
AK high	1–1

In several California card rooms, the payoff odds on the Call bet can be very different from Table 7.14. Some card rooms sharply reduce payoffs at the high end and slightly increase them for some midrange hands. Three pay tables are shown in Table 7.15.

Some online casinos use Table 7.16 (page 225) for the Call bet. Though Tables 7.14–7.16 look very different, the expected returns among them, assuming a winning player hand, only differ by about 2¢ per dollar. The most striking differences come in the payoff odds for royal and straight flushes. Since these hands are very rare, a large or small payoff barely affects the overall expectation. Subsequent analysis will assume that Table 7.14 is in use for the Call bet.

TABLE 7.15: Caribbean stud poker: Call bet payoffs in California card rooms.

	Payoff schedule		
Player hand	**A**	**B**	**C**
Royal flush	10–1	200–1	10–1
Straight flush	9–1	50–1	9–1
Four of a kind	8–1	30–1	8–1
Full house	7–1	7–1	7–1
Flush	6–1	5–1	6–1
Straight	4–1	4–1	5–1
Three of a kind	3–1	3–1	3–1
Two pairs	2–1	2–1	2–1
One pair	1–1	1–1	1–1
AK high	1–1	1–1	1–1

TABLE 7.16: Caribbean stud poker: Call bet payoffs in online casinos.

Player hand	**Payoff odds**
Royal flush	800–1
Straight flush	200–1
Four of a kind	25–1
Full house	10–1
Flush	7–1
Straight	5–1
Three of a kind	3–1
Two pairs	2–1
One pair	1–1
AK high	1–1

Example 7.13. Continuing with Example 7.12: If you make the Call bet and the dealer beats your hand, which is relatively low, you lose \$3, and if you fold instead of calling, you lose \$1. Assume that you make the Call bet. If the dealer fails to qualify, you will win \$1 on your Ante bet, and the Call bet is a push. If the dealer qualifies, you win \$1 on the Call bet and \$1 on the Ante if his or her hand cannot beat yours.

The dealer's hand turns out to be

$$K\spadesuit\ Q\clubsuit\ J\heartsuit\ 8\heartsuit\ 2\heartsuit,$$

so the dealer fails to qualify. You win \$1 from your Ante bet. Though your hand beat the dealer's, your Call bet has no action since the dealer did not have at least AK. ■

As an even-up game would hold no advantage for a casino offering CSP, the rules have been set to give the house an advantage. There are two built-in rules by which the casino gains an edge over the players:

- One exposed dealer card doesn't necessarily convey useful information, and so players might fold winning hands based on the strength of the dealer's exposed card. For example, if the dealer shows an ace, a player might fold a low pair or ace-king hand that could beat a qualifying dealer hand with no pair or could win the Ante bet if the dealer fails to qualify even with that ace.
- By requiring that the dealer have a qualifying hand, some potential high-paying player hands win only even money on the Ante bet if the dealer doesn't qualify. A player making a $10 Ante bet and backing it up with a $20 Call bet upon being dealt a flush, for example, wins only $10 instead of $110 if the dealer's hand doesn't qualify. If you beat the low probability of .198% by receiving a dealt flush, you want to cash in for as much as you can, but the requirement that the dealer qualify takes some of that opportunity away.

The second rule shows the true challenge facing a CSP player. You lose if the dealer's hand is good enough to beat yours, as in traditional 5-card stud poker, but you also lose money, in the sense of winning less money than the laws of probability suggest you ought to, if the dealer has a bad hand.

Strategy

Since there is a player choice in CSP, the minimum house advantage is calculated assuming perfect play, and starts at 5.22% [58]. Imperfect play, of course, causes this advantage to rise.

There are 1,135,260 possible dealer hands that do not qualify [64], so the probability of a dealer qualifying hand is

$$\frac{1,463,700}{2,598,960} \approx .5632.$$

Combining this with the known probabilities for poker hands (Table 1.2) leads to an interesting observation: even if you are dealt a rare high-ranking hand, the chance that the dealer will fail to qualify and you will only collect on your Ante bet is 43.68%, about 3 hands out of 7.

With this probability in mind, we can develop an optimal player strategy that takes the form of "Fold all hands ranking lower than X and call with any hand that beats X". This hand X, at the boundary between calling and folding, is called the *beacon hand*.

A very simple plan, one that barely counts as a strategy, is to call on every hand, no matter what. This takes the element of skill completely out of the game and reduces CSP to a game of chance. You'll be very popular with the casino—until your bankroll runs out. While this plan is superficially attractive in that it requires no thought and you'll win 1 unit every time the dealer fails to qualify, it misses the fact that you'll lose a lot of 3-unit bets when you call on very weak hands and the dealer qualifies, in addition to the standard 5.22% loss that arises from perfect play. The house edge using a "call on everything" strategy rises to 16.61%—more than triple the HA from perfect strategy [132].

As we look for a plan with some nuance to it, we can easily see that a player should make the Call bet on any hand containing a pair or better, even allowing for the possibility that a qualifying dealer hand may beat it. Losses are limited by the amount of the Ante and Call bets, while a player may win more than his or her wager with a good hand. Along the way, the strategy should also take into account the information provided by the dealer's upcard—if an ace is showing, it is somewhat more likely that the dealer will qualify than if the upcard is a 6.

The challenging part of developing a strategy comes when considering the Caribbean stud poker equivalent of bluffing: making the Call bet on a weak hand. This is an option that might be taken if the dealer shows a low upcard and is less likely to qualify, which means that the player will be paid even money on the Ante bet despite a weak hand—possibly one that loses to the dealer's nonqualifying hand.

A detailed analysis of CSP was published in 2000 [58], where three increasingly more complicated strategies are presented.

1. The beacon hand is $AKJ83$. Call with any hand of $AKJ83$ or better, and fold otherwise.

2. Strategy #1 with 618 exceptions, all involving player AK hands and the rank of the upcard. These exceptions can be sorted into 4 broad groups that are easier to learn. This strategy calls for players to make the Call bet on any $AKQJx$ hand and to incorporate the upcard into other decisions. For example, the hand $AKT76$ should be called, instead of folded as in strategy #1, if the upcard is a T, 7, or 6, since the probability that the dealer pairs her upcard is lower due to the presence of a T, a 7, and a 6 in the player's hand.

3. Strategy #2 with 105 further exceptions that consider the number of player cards of the same suit as the upcard.

Example 7.14. Consider the player hand $AKQ85$.

- Strategy #1 states that a player should call on this hand regardless of the dealer's upcard.

- Strategy #2 calls for players to fold this hand against a dealer jack, 10, or 9, since the player cannot assist in blocking a dealer pair of those.

- When using strategy #3, a player should make the Call bet on this hand against a jack if the player has 4 cards of the jack's suit. Strategy #2 also directs the player to call on $AKQ85$ against a dealer 7 or 6, but strategy #3 favors folding against a 7 or 6 if no cards have the same suit as the upcard. In this scenario, the probability that the dealer's hand is a flush is

$$\frac{\binom{12}{4}}{\binom{46}{4}} = \frac{3}{989} \approx .0030,$$

a small chance, perhaps, but at this level of detail, the importance of small probabilities is magnified.

■

While the first strategy is simple to use, strategies 2 and 3 pose quite a challenge to memorize, and the exceptions in strategy #3 are probably too numerous to print them on a wallet-sized card for use at the tables, as is often done with blackjack basic strategy. How much of an advantage do the more refined strategies give?

The house edges for the 3 strategies described above are listed in Table 7.17.

TABLE 7.17: House edge for 3 Caribbean Stud Poker strategies [58].

Strategy	**HA**
1	5.316325%
2	5.224385%
3	5.224305%

Switching from the first "call on $AKJ83$ strategy to the second gives an extra .09% to the player (9¢ per $100 wagered), which might just barely motivate a committed player to memorize the exceptions. The difference between strategies 2 and 3 is .000079%, amounting to .0079¢ per dollar wagered.

We shall examine some of the mathematics that goes into determining the beacon hand $AKJ83$. Consider first the expectation of a $1 ante bet when we call despite holding a hand lower than AK, which will lose the Call bet whenever the dealer qualifies. We know that if we fold a hand, our expected return is –$1; if a particular hand has an expected return greater than this, we should make the Call bet. Just as the dealer has approximately a 44% chance of not qualifying, the probability that our hand is less than AK is also .4368. Under these circumstances, we lose $3 every time we call and the dealer qualifies. If the dealer doesn't qualify, we win $1 each time regardless of the relative value of the two hands. It follows that the expected value of calling on a hand below AK is

$$E = (-3) \cdot \left(\frac{1,463,700}{2,598,960}\right) + (1) \cdot \left(\frac{1,135,260}{2,598,960}\right) = -\frac{114}{91} \approx -\$1.25,$$

which is worse than simply folding and taking the \$1 loss.

However, if our hand contains an ace and a king but no pair, and thus has the potential to beat some qualifying dealer hands, then our expectation is higher than –\$1.25 and is a function of the strength of the hand. The challenge comes in determining the cutoff hand where the hand's expected value E switches from $E \leqslant -1$ to $E > -1$. We begin by considering the best no-pair scenario. Consider the highest possible hand without at least a pair: a no-flush $AKQJ9$. This will beat any qualifying dealer AK hand; we ignore the very small probability of a push, which depends on the suits present in the hand due to the need to consider flushes, and is less than 1 chance in 10,000. Since there are 167,820 five-card AK hands that are not straights or flushes [64, p. 326], it follows that the expectation when making a Call bet on $AKQJ9$ is

$$\begin{aligned} E &= (-3) \cdot \left(\frac{1,295,880}{2,598,960}\right) + (3) \cdot \left(\frac{167,820}{2,598,960}\right) + (1) \cdot \left(\frac{1,135,260}{2,598,960}\right) \\ &= -\frac{18,741}{21,658} \\ &\approx -\$.865. \end{aligned}$$

Since the expected value of –\$.865 is greater than –\$1, it follows that if you are dealt $AKQJ9$, the Call bet is in order. The same holds true—as we guessed above—for any hand that beats $AKQJ9$.

At this point, our CSP strategy looks like this:

- Call with any hand ranked $AKQJ9$ or higher. This includes all hands that contain a pair or higher, and thus have the potential to beat a qualifying dealer hand.

- Fold any hand not holding at least an ace and a king. These are the hands that cannot beat a qualifying hand, and thus have an expectation of –\$1 if folded vs. –\$1.25 if called.

This covers over 93.4% of all possible dealt hands and gives a house edge of about 5.5%; the AK hands are all that remain. Thinking of these hands as arranged in descending order from $AKQJ9$ to $AK432$, we note that if we raise on $AKQJ9$, we gain about 13% relative to folding, whereas if we raise with $AK432$, we would lose about 25% versus folding [64]. The conclusion is that the beacon hand X mentioned above is about $\frac{2}{3}$ of the way up from the bottom of the list of AK hands, which turns out to be $AKJ82$.

Note that at this point, with most of a strategy determined, this analysis has not yet made use of the dealer's upcard—it's not been necessary. This information is incorporated as we refine these rules into a final strategy, which concludes with the following rules determined with an eye on the possible number of dealer flushes:

- Call on any hand that beats $AKJ82$ and fold any hand that doesn't. This is strategy #1 from the list on page 227.
- Call on $AKJ82$ if your hand includes all 4 suits, otherwise fold [64]. If the hand contains all 4 suits, the probability of a dealer flush drops. A hand with suits distributed $a - b - c - d$, where $a + b + c + d = 5$, has

$$\binom{13-a}{5} + \binom{13-b}{5} + \binom{13-c}{5} + \binom{13-d}{5}$$

possible dealer flushes, with the upcard not considered. Table 7.18 shows the number of possible dealer flushes for each distribution of 5 player cards among the 4 suits.

TABLE 7.18: Caribbean stud poker: Possible dealer flushes for different player suit holdings.

Player suit distribution	**Possible dealer flushes**
2–1–1–1	2838
2–2–1–0	3003
3–1–1–0	3123
3–2–0–0	3288
4–1–0–0	3492

In addition to the upcard, any information about the cards held by the other players can be of some use to a CSP player. Since these cards cannot be among the dealer's hole cards, their presence in players' hands may provide important information about the dealer's chance of qualifying. Moreover, since players are not in competition with one another, there is nothing lost if another player knows the cards in your hand.

Example 7.15. Suppose that you are playing CSP with three other gamblers. You hold $A\spadesuit\ J\diamondsuit\ 7\clubsuit\ 5\spadesuit\ 2\heartsuit$, which would not merit a Call bet by our strategy above, and the dealer's upcard is the $Q\clubsuit$. If you are informed that the other three queens are all contained in your fellow gamblers' hands, you know that the dealer cannot hold a pair of queens, and thus the chance of a qualifying hand is diminished. It may then be mathematically sound to make the Call bet in the hopes of winning your Ante bet when the dealer fails to qualify. ■

For this reason, casinos are ordinarily vigilant about not allowing players to communicate the contents of their hands to each other. Many casinos expressly forbid communication between players about their hands, but in the summer of 2020 with the Covid-19 pandemic in full force, some casinos, hoping to stay open while also respecting calls for social distancing, considered dealing card games face up in order to minimize player contact with the cards which might spread the virus. The state of Washington carefully considered the implications of dealing cards face up on the casino's advantage in CSP [60]. There are two types of cards to consider: cards which match the dealer's upcard and so affect the chance of a pair, and aces and kings, which affect whether or not the dealer can qualify with an AK hand.

Analysis of the effects of players revealing information about their holdings of either type of card showed that the casinos' 5.32% HA could be turned into a 1.15% player advantage provided that there were 7 players each signaling their hands [60, 77]. This player edge could be achieved with some changes in betting strategy [77]:

- The beacon hand $AKJ83$ should be folded unless the dealer cannot have a pair because all 3 cards of the upcard's rank are held by players.

  $AKJ83$ without a flush is a rare enough hand (1 hand in 2548) to begin with that this tip poses a very limited threat to the casino.

- A low-pair hand (2s through 6s) should be folded if the upcard is higher than the player's pair, unless at least 2 upcard copies are with the players.

- A high pair (7s through Ks) should be folded against a higher upcard unless at least 1 copy of the upcard is out.

- If the other players hold all 3 cards matching the dealer upcard and 4 or more aces or kings, the player should make the Call bet on any hand. This rule covers the case in Example 7.15, provided that the other players hold at least 3 more aces or kings.

  Assume that there are n players and that the upcard is neither an ace nor a king. The probability of meeting these conditions is

$$P_n = \frac{\binom{3}{3} \cdot \sum_{k=4}^{8} \left[\binom{8}{k} \cdot \binom{40}{5n-k-3} \right]}{\binom{51}{5n}}.$$

  P_n is tabulated in Table 7.19.

TABLE 7.19: Caribbean stud poker: Probability of useful information from n player hands in collaboration.

n	P_n
3	.0021
4	.0158
5	.0605
6	.1526
7	.2926

> The small value of P_3 shown here, combined with the limited scope of player hands for which this knowledge might affect player strategy, led the Washington authorities to approve dealing CSP face up to 3-player tables. With tables limited to 3 players due to the pandemic, it was felt that dealing the cards face up, giving each player information about 10 cards rather than 30, would not result in a player edge. Three-player tables allowed for more physical distance between players, and so served as an additional safeguard against Covid.

It was good to be early. Caribbean Stud Poker is credited for starting a growth industry in new casino table games. CSP had a good run among carnival games, with over 1500 tables in operation at one time, but declined rapidly. Fewer than 100 tables are still running. Part of that decline is due to the loss of the novelty factor as new poker-based carnival games have been introduced, but part is also due to the high chance of a dealer nonqualifying hand [44]. If a player dealt a straight is forced to wager $15 and has approximately a 44% chance of only winning $5 instead of $45 on this rare but lucrative hand, that's a player who may well decide to play a different game.

A CSP player who is dismayed by the possibility of losing a lucrative payoff on a high hand when the dealer fails to qualify might be attracted to the optional Bonus Insurance bet offered at the Commerce Casino in Commerce, California. This bet wins if the dealer fails to qualify and has 2 pay schedules: one that pays off on a strong player hand and one whose payoff odds are based on the dealer's hand. Only the higher payoff is made on a winning hand; Table 7.20 shows the payoffs.

TABLE 7.20: Caribbean stud poker: Bonus Insurance pay table.

Player Hand	**Payoff**
Royal flush	100–1
Straight flush	50–1
Four of a kind	20–1
Full house	10–1
Flush	7–1
Dealer Hand	**Payoff**
7 high	6–1
8 high	3–1
9 or 10 high	2–1
Lower than AK high	1–1

Players wishing to make the insurance bet must do so before the cards are dealt. If the dealer's upcard is a 2–5, players have the option to double their insurance wager. A player holding a flush or higher will certainly double an insurance bet when possible; this is as close to free money as a casino ever offers.

Wild Aruba Stud

When it was introduced in the late 1990s, *Wild Aruba Stud* was advertised as "Basically Caribbean Stud with Deuces Wild". Introducing wild cards, as we have seen, leads to generally higher hands, and so the dealer's qualifying hand in Wild Aruba Stud rises to a pair of 8s.

If the dealer's upcard is a deuce, then she will always make a qualifying hand. Any 4 cards less than an 8 must necessarily form a straight with the wild deuce.

There are 1,225,008 one-pair hands when deuces are wild, but unlike in the standard deck, they are not evenly distributed among the 12 remaining ranks. Indeed, the only hands containing a pair of 7s or lower are natural pairs with no deuces in the hand. These hands number

$$5 \cdot \binom{4}{2} \cdot \binom{11}{3} \cdot 4^3 = 316,800.$$

The probability that the dealer qualifies is

$$\frac{1,482,480}{2,598,960} \approx 57.04\%,$$

a slight increase from the 56.32% chance that a CSP dealer qualifies.

Let It Ride

Deck composition:	52 cards
Hand size:	5 cards
Community cards:	2
Betting rounds:	3
Required bets:	$
Optional bets:	1, 2

In *Let It Ride*, betting works in reverse. Where a standard stud poker player bets more as cards are dealt and his or her hand improves, Let It Ride players have the option of withdrawing bets already made as the cards come out if their hand doesn't get better. Let It Ride was invented in 1993 by John Breeding, the founder of Shuffle Master, a manufacturer of automatic card shuffling machines. The game was designed to create a new market for Shuffle Master's product [4].

The object of Let It Ride is to make the best possible five-card poker hand from three personal cards and two community cards. As with games like Dakota Duel Draw, the dealer does not receive a separate hand. Winning hands—anything that's at least a pair of 10s—are paid off according to a fixed pay table; there is no dealer hand to beat. Payoff tables vary among casinos; one common set of payoffs, and the one used at the Soaring Eagle Casino in Mount Pleasant, Michigan, may be found in Table 7.21.

TABLE 7.21: Let It Ride payoffs [4].

Player hand	**Payoff odds**
Royal flush	1000–1
Straight flush	200–1
Four of a kind	50–1
Full house	11–1
Flush	8–1
Straight	5–1
Three of a kind	3–1
Two pairs	2–1
Pair of 10s or higher	1–1

Players begin a hand of Let It Ride by making three bets of equal amounts, in spaces labeled 1, 2, and \$. The "\$" bet may never be withdrawn, and remains in play until the hand is complete. Three cards are dealt face down to each player, and the dealer receives two cards dealt face down, which will serve as community cards for all players, as in Texas hold'em. Upon examining their first three cards, the players may withdraw the "1" bet.

The dealer then exposes one of the community cards, and again, players may pull back one bet, this time the "2" bet. Finally, the second community card is revealed, and at this point, all remaining bets are resolved and paid off in accordance with the payoff table. If the player's hand does not contain at least a pair of 10s, then all bets left on the board lose and are collected.

Example 7.16. In a two-player game of Let It Ride, Sandy is dealt 8♣ 5♡ 4♢ and Kim is dealt Q♠ 7♠ 3♢. Sandy, holding 3 low cards of different suits, opts to pull back bet 1. Kim's makes the same choice despite holding a queen.

The first community card is the Q♡, pairing Kim's Q♠ and delivering a guaranteed winner, so Kim lets bet 2 ride. Sandy withdraws bet 2, since the hand has little potential to improve to a pair fo 10s or better,

The second community card is the T♢, giving Sandy a losing queen-high hand (that only lost one bet due to smart withdrawal of the optional bets) and Kim a pair of queens paying 1–1 on both remaining bets. ■

The calculated house advantage for Let It Ride, 3.50%, is based on perfect player strategy, and any deviation from that strategy serves to increase the HA. How should a player make those perfect decisions about whether or not to let bets 1 and 2 ride? There are a couple of obvious informal rules to follow: if you are dealt at least a pair of 10's, let them ride, but if your first four cards (including the first community card) are all 9s or lower, pull back bet 2. Indeed, pulling back bet 1 if you are dealt three nonsuited low cards is probably advisable, as we saw in Example 7.16.

What about letting bets ride on three or four cards to a straight or flush, which is a non-winning hand as it stands but has potential for a high payoff?

As a general rule, this is inadvisable, as the following strategy will show. Once again, computer simulation of many hands leads us in the best possible direction. By playing according to the following strategy, you can cut the house advantage to 3.50% [4].

- Pull back bet 1 unless you hold one of the following:
  - Any winning hand (a pair of 10s or higher).
  - Three cards to a royal flush.
  - Three cards to a straight flush with the lowest card at least a 3.

  The restriction on this third option is to allow the straight to fill from either end—a hand such as 2♡ 3♡ 4♡ can only be completed to a straight in two ways (A–5 and 5–6), whereas 3♡ 4♡ 5♡ admits three paths to a straight (A–2, 2–6, and 6–7).

- After the first community card is exposed, pull back bet 2 unless you hold one of the following:
  - Any winning hand.
  - Four cards to a flush (including a royal flush).
  - Four cards to an open-ended straight. This is a straight that does not include an ace, and thus can be completed by two different ranks of card.
  - Four cards to an inside straight, provided that the lowest card is at least a 10—for example, any $TQKA$. While the inside straight is less likely to complete on the fifth card, the possibility of pairing one of your four cards to a winning pair makes up for the decreased probability of finishing the straight.

This strategy suggests rather clearly that the majority of starting hands do not merit letting the retractable bets ride. Low pairs do not justify a bet, nor do single high cards or three-card straights or flushes. Since you can never withdraw the $ bet, you are still in a position to win some money if the second community card, for example, completes a hand consisting of a pair of 7s and a 9 to two pairs or three of a kind.

Example 7.17. Suppose you are dealt J♣ Q♣ 5♡ playing one-on-one against the dealer. What is the probability that the two community cards will give you a winning hand?

There are two ways that you can win with this starting hand:

1. Pull additional jacks or queens in the community cards.
2. Find a pair of 5s, 10s, kings, or aces in the community cards.

In case 1, you need to pair either the jack or queen, and you have two cards to do so. Six cards will give you a winning combination. There are

$$6 \cdot 44 + \binom{6}{2} = 279$$

ways for that to happen. In this total, the first term includes pairing the jack or the queen individually and includes the case of drawing a jack or queen with an additional 5, and the second covers the case where the community cards include two jacks, two queens, or one of each, which complete your hand to two pairs or three of a kind.

For case 2, there are $\binom{4}{2} = 6$ ways to draw two of any rank, thus 24 ways for the community card to form a pair that gives you a winning hand: either three 5s or a pair of 10s, kings, or aces.

Putting everything together gives the following probability of winning:

$$\frac{279 + 24}{\binom{49}{2}} = \frac{303}{1176} \approx .2577,$$

which is not enough to justify risking two bets.

■

Big Raise Poker

Deck composition:	52 cards
Hand size:	5 cards
Community cards:	2
Betting rounds:	2
Required bets:	Ante
Optional bets:	Play

Big Raise Poker follows a familiar pattern: players make an Ante bet, are dealt some cards (3 here), and then have the chance to fold or back up their ante with a Play bet before the remainder of the hand—in this case, 2 community cards—is dealt. In Big Raise Poker, the Play bet is fixed at 4 times the Ante bet, which suggests a "go big or go home" strategy.

In part, perhaps, to compensate for the increased risk assumed by the player due to the size of the Play bet, the pay table extends as low as a pair of 6s; it is shown in Table 7.22.

From Tables 1.2 and 7.22, we can show that the house has a 25.94% edge in the game. If a player folds his or her first 3 cards, the return is –\$1; playing through to the end and wagering \$5 overall gives an expected value of –\$1.30. This suggests that folding every hand is the correct decision—which is patently

TABLE 7.22: Big Raise Poker pay table [109].

Hand	**Payoff**
Royal flush	500–1
Straight flush	100–1
Four of a kind	40–1
Full house	8–1
Flush	6–1
Straight	4–1
Three of a kind	3–1
Two pair	2–1
High pair (10–A)	1–1
Medium pair (6–9)	Push

absurd, since a hand containing a medium pair, high pair, or 3 of a kind is guaranteed not to lose money and so should be backed with the 4× Play bet.

That being so, we need a different path to a Big Raise Poker strategy. As in several other games, it is possible that a hand with good potential for improvement might have an expected value of more than –\$1, and so the gambler holding it should make the Play bet.

Example 7.18. Consider the 3-card hand $\diamondsuit QJT$, which is 3 cards to a royal flush, and also has the possibility of improving to several other paying hands, all the way down to a medium pair. There are

$$\binom{49}{2} = 1176$$

ways to draw the 2 community cards. Of these:

- 1 leads to a royal flush.
- 2, $\diamondsuit K9$ and $\diamondsuit 98$, give a straight flush.
- There are no combinations that result in 4 or a kind or a full house, both of which require a pair in hand.
- A flush occurs in

$$\binom{10}{2} - 3 = 42$$

  ways, where the straight and royal flush draws are subtracted.
- There are 3 ways to draw 2 ranks that complete a straight: $AK, K9$, and 98. Each contributes $4 \cdot 4 - 1$ ways to pick the suits to form a flush that is not a straight or royal flush, for a total of 45 straights.
- 3 of a kind requires a pair of queens, jacks, or 10s in the community cards. There are 9 ways to draw one of these pairs.
- A 2-pair hand is formed when the community cards are QJ, QT, or JT. Each card permits 3 choices, for a total of 27 2-pair draws.

- High pairs are formed when the draw is AA, KK, Qx, Jx, or Tx, where x is any rank other than Q, J, or T. 372 pairs of community cards complete a high pair.
- A medium pair must be drawn as a pair of 6s through 9s, and this may be done in 24 ways.
- The remaining 654 choices for the community cards—over 50% of the total—lead to a losing hand.

The expected return if the 4× Play bet is made is \$3.96, a positive value which justifies a Play bet when holding 3 cards to a royal flush. \$2.13 of this comes from the \$2500 payoff on a royal flush, which has probability $\frac{1}{1176}$. ■

That 3 cards to a royal flush is worth a raise is not surprising. It is instructive to consider hands that might lie closer to the border between raising and folding. Consider the lowest possible low-pair hand: 223. Since the hand already contains a low pair, it cannot be completed to a flush, straight, high pair, middle pair, or high card hand. Table 7.23 shows the possible outcomes if the player makes the 4× Play bet and sees the community cards.

TABLE 7.23: Big Raise Poker: Outcomes when drawing to 223.

Hand	**Ways to draw**
4 of a kind	1
Full house	9
3 of a kind	88
2 pairs	198
Low pair	880

The expected return when 223 is backed by a Play bet is –\$0.46. The house still has an advantage, but it's smaller: 9.18%. The precise card ranks 2 and 3 do not affect the calculation of the expected value here; this is the expected return on *any* low-pair hand with a third low card. Since this is greater than the –\$1 return from folding, it follows that any low pair—and therefore any pair—should be played.

Hickok's Six Card

Deck composition:	52 cards
Hand size:	5 or 6 cards
Community cards:	2 or 3
Betting rounds:	2
Required bets:	Ante
Optional bets:	Double, Buy

One of the most enduring poker legends centers on Old West folk hero Wild Bill Hickok. Hickok was murdered on August 2, 1876 while playing draw poker in Deadwood, in present-day South Dakota. He was holding two pairs, black aces and 8s, when he was shot. As a result, the hand $A\spadesuit\ A\clubsuit\ 8\spadesuit\ 8\clubsuit$ has come to be known as the *dead man's hand* (Figure 7.2). Sources do not agree

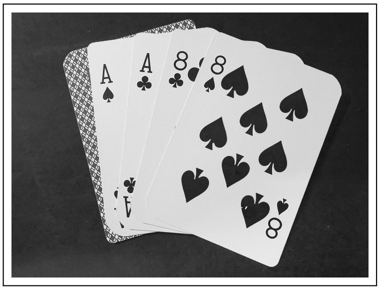

FIGURE 7.2: Dead man's hand, held by Wild Bill Hickok when he was murdered. The fifth card is unknown.

on what the fifth card in Hickok's hand was; one story asserts that Hickok had not picked up the 5th card when he was shot and it was scattered in the mayhem that ensued [154].

Hickok was inducted as a member of the inaugural class of the Poker Hall of Fame in 1979 [154]. His name was attached to a new poker game in 1993, when Casino Magic in Bay St. Louis, Mississippi and Foxwoods Casino in Mashantucket, Connecticut launched *Hickok's Six Card*: a carnival game which was novel in allowing players to pay an extra fee to draw an additional community card.

Play in a hand of Hickok's Six Card proceeds as follows [153]:

- Players make an initial Ante bet and are dealt 3 cards each. The dealer does not take a hand.
- Based on this 3-card hand, players may fold and surrender half of their Ante, play on with no additional wager, or double their bet.
- Two community cards are dealt to the table.
- Players may stand on their 5-card hand or may pay a fee of half their Ante to share in a third community card. This fee receives no action.
- Players who do not pay the fee are paid off according to Table 7.24. The minimum winning hand in this game is a pair of jacks, although a pair of jacks, queens, or kings merely pushes.
- The third community card is dealt. All remaining players make their best 5-card hand from their 3-card hand and the three community cards, which is also paid off following Table 7.24.

TABLE 7.24: Hickok's Six Card pay table [153].

Hand	**Payoff**
Royal flush	200–1
Straight flush	50–1
Four of a kind	20–1
Full house	8–1
Flush	5–1
Straight	4–1
Three of a kind	3–1
Two pair	2–1
Pair of aces	1–1
Pair of face cards	Push

Example 7.19. Suppose that you hold J♠ J♢ 7♣ 5♣ 3♠. This 5-card hand is a push as it stands. Is it worth it to buy into the third community card?

Without buying the 6th card, a $1 bet on this hand has an expected value of 0. If you pay the fee of 50¢, then the outcomes of your hand are shown in Table 7.25.

TABLE 7.25: Hickok's Six Card: Outcomes when taking a 6th card to J♠ J♢ 7♣ 5♣ 3♠.

Final Hand	**Probability**
3 jacks	$^2/_{47}$
Two pairs	$^9/_{47}$
Pair of jacks	$^{36}/_{47}$

The expected value of the enhanced hand is

$$E = (3) \cdot \frac{2}{47} + (2) \cdot \frac{9}{47} + (0) \cdot \frac{36}{47} - .50 \approx -\$.1383.$$

Since this is less than the value of the original 5-card hand, buying the 6th card is not advised.

If, however, you doubled your original bet to $2 after seeing your first 3 cards, the fee for the third community card remains half your initial bet: 50¢. The expected value in this case is

$$E = (6) \cdot \frac{2}{47} + (4) \cdot \frac{9}{47} + (0) \cdot \frac{36}{47} - .50 \approx \$.5213,$$

a positive value which favors buying the sixth card.

Doubling your ante on this hand is called for precisely if your first 3 cards include both of the jacks. At this point, your hand cannot lose money no matter which community cards appear, so putting more money on the table to increase the payout if your hand improves in the community cards carries no risk. ■

Which 3-card hands merit a raise? It's clear that any pair of face cards, pair of aces, or three of a kind should be immediately doubled, but what about a hand which is not itself a winner but contains great potential for improvement, such as $J\spadesuit\ T\spadesuit\ 9\spadesuit$: 3 cards to an open-ended straight flush?

The PDF for the possible outcomes when the community cards are drawn to this hand are shown in Table 7.26. The denominator, 1176, is $\binom{49}{2}$.

TABLE 7.26: Hickok's Six Card: Outcomes when drawing 2 community cards to $J\spadesuit\ T\spadesuit\ 9\spadesuit$.

Final Hand	**Probability**
Straight flush	3/1176
Flush	42/1176
Straight	45/1176
3 of a kind	9/1176
Two pairs	27/1176
Pair of aces	6/1176
Pair of face cards	132/1176
Losing hand	549/1176

Note that the probability of a losing hand is less than ½. The expected value of this 3-card hand is positive, so doubling the bet is the best decision.

That having been said, what is the best choice if the hand improves, but not to a paying hand, with the first 2 community cards? Suppose that the community cards are $A\spadesuit\ T\diamondsuit$. The straight flush is no longer possible, but a 6th card might yet elevate this 4-card flush with a low pair to a winner.

There are 2 options: fold the hand and incur a loss of \$2 (since the wager was properly doubled), or pay the 50¢ fee to buy a sixth card. If you buy the extra card, your hand can improve to a flush, 3 tens, or 2 pairs. The probability of improving is $\frac{20}{47}$, and the expected value of the hand, accounting for the doubled bet and the cost of buying the extra community card, is \$1.29. You should play on.

An example of a hand that should be folded is $T\clubsuit\ 6\diamondsuit\ 2\diamondsuit$: three low unsuited cards without a 2-card straight draw. Folding before the community cards are dealt incurs a loss of 50¢; playing on leaves a hand that can rise no higher than 3 of a kind and has an expected value of about –92¢. The player incurs a smaller loss by folding.

7.3 Division Carnival Games

Pai Gow Poker

Deck composition:	53 cards
Hand size:	7 cards
Community cards:	None
Betting rounds:	1

Pai Gow poker is a division game that uses seven cards per player and whose roots are found in *pai gow*, a domino-like game that is said to flourish in underground Chinese casinos. The best hand in pai gow is a 9, and so the name of the game is Cantonese for "make nine". In pai gow poker, "make nine" is sometimes taken in the opposite way: to express a wish that the dealer will have the worst possible hand: 2345789 with no flush. An unusual feature of pai gow poker is that players can serve as the banker and cover all other bets, including a bet against the casino's hand. All that is necessary is that the player is willing to bank and have enough money to pay off all other bets.

Pai gow poker uses a 53-card deck, including a single joker used as a bug. The game is played against a dealer hand. Players and the dealer are each dealt seven cards, which then must be "set," or sorted, into a five-card poker hand, the *back* hand, and a two-card hand, the *front* hand. It is a requirement of the game that the five-card hand be better than the two-card hand, as is the case in Chinese Poker.

The back hand uses standard poker hand rankings. Five aces is the highest hand, beating a royal flush. In most casinos, a wheel is the highest straight flush and second highest straight, losing only to $AKQJT$.

For the front hand, the highest hand is a pair of aces, followed by lower pairs and then by high-card hands. Straights and flushes have no standing in the front hand.

Example 7.20. Suppose you are dealt the following hand:

$$K\clubsuit\ Q\spadesuit\ Q\diamondsuit\ Q\clubsuit\ 7\clubsuit\ 6\heartsuit\ 4\clubsuit.$$

This hand can be split into three queens and $K\clubsuit\ 7\clubsuit$, thus meeting the requirement that the back hand outrank the front hand. It could also be split into a pair of queens and KQ. ■

Players make a single wager, which pays off at 19 to 20 if both player hands beat the corresponding dealer hands. The 19–20 payoff represents a 5% casino commission charged on winning hands, and is the source of the house advantage. If one player hand beats the dealer and the other doesn't, the hand is a push. Ties on either hand, known as "copies", go to the bank, and hence the player loses. It is in these two ways that the casino derives its advantage, which comes to 1.57% from charging commissions and 1.27% from taking all ties, a total of 2.84% [69, p. 149–50].

The dealer has a fixed strategy, the *house way*, when setting his or her hands. This is similar to the rules guiding blackjack dealers: their actions are prescribed in advance and are independent of the players' hands.

Example 7.21. While the house way might vary slightly among casinos, the following rules are fairly standard.

- With **1 pair**, place the pair in the back hand and the next two highest cards in the front hand.
- When holding **2 pairs**: Split the pairs if holding a pair of aces, or if holding a pair of face cards and 6s or higher. With two pairs of 6s or less, put the two pairs in the back hand. With any other two pairs, split them unless the hand contains an ace, then play the ace in the front hand.
- If dealt **2 three of a kinds**, play the lower three of a kind in the back hand and split the higher three of a kind.

■

The house way is also the optimal strategy for players. While the particulars of the house way may vary slightly from casino to casino, the small differences have little effect on the overall house edge.

Inherent in the house way and the play of pai gow poker hands is that making decisions about how to sort one's seven cards into front and back hands takes time, and so the game proceeds somewhat more leisurely than other table games such as blackjack and baccarat. While the edge is still with the casino, the diminished game pace makes for fewer decisions per hour, and thus a pai gow poker player will lose money more slowly in the long run than a gambler at quicker games with comparable HAs.

Example 7.22. If the dealer holds the following hand:

$$5\clubsuit\ 6\heartsuit\ 5\diamondsuit\ 6\spadesuit\ 9\heartsuit\ Q\spadesuit\ J\clubsuit,$$

then the house way would keep the two pairs together since they're both 6s or lower, and then make the front hand as high as possible, so that the front hand would be $Q\spadesuit\ J\clubsuit$ and the back hand would be $5\clubsuit\ 5\diamondsuit\ 6\heartsuit\ 6\spadesuit\ 9\heartsuit$. ■

Example 7.23. For the hand

$$J\heartsuit\ 9\diamondsuit\ 7\spadesuit\ K\heartsuit\ A\diamondsuit\ Q\diamondsuit\ 5\heartsuit,$$

the best that the dealer can do is to make two high-card hands. Remembering that the back hand must outrank the front hand, the ace must go into the front hand. That having been done, it makes the most sense to put the king and queen into the back hand, making it as high as possible in the hopes of catching some low two-card player hands. ■

What is the probability of being dealt a hand including the joker?

There are $\binom{53}{7} = 154,143,080$ different seven-card hands that can be dealt. Of these, $\binom{52}{6} = 20,358,520$ of them contain the joker—we simply count the number of ways to draw the remaining six cards from the 52 non-jokers. Accordingly, the probability of receiving the joker is

$$p = \frac{20,358,520}{154,143,080} = \frac{7}{53} \approx .1321,$$

so the joker appears slightly less than once every seven hands.

Owing to the restrictions on how a wild joker may be used, there is only one five-of-a-kind hand: five aces. The probability of being dealt five aces is

$$p = \frac{\binom{48}{2}}{\binom{53}{7}} = \frac{1128}{154,143,080} \approx 7.318 \times 10^{-6},$$

where the numerator simply counts the number of ways to choose the other two cards from the 48 deuces through kings.

It's extremely unlikely that a skilled player who is dealt five aces will put them in the five-card hand. The house way stipulates that five aces should be split unless the other two cards are a pair of kings. This is easy to understand: a pair of aces is the best possible front hand, and moreover cannot be beaten or tied by the dealer in this case, since the player holds all of the other aces. Holding all of the aces also means that the dealer cannot beat a KK front hand. Three aces is a strong back hand, and thus very likely to win without additional value.

A hand that is available in pai gow poker but not in any five-card poker game is two three-of-a-kind hands—though it cannot be played that way when the cards are divided. What is the probability of being dealt this hand?

We must split this into two cases: hands with and without the bug being used as a third ace. If you do not hold three aces, then the number of hands with two three of a kinds is

$$N_1 = \binom{12}{2} \cdot \binom{4}{3}^2 \cdot 45 = 47,520.$$

In this expression, the first factor is the number of ways to pick two ranks that are not aces, the second counts the ways to select the cards, and the third accounts for the seventh, non-matching card. The joker may be drawn as card #7, since with two sets of three matching non-aces, it will not be used to fill out a straight or flush, and so will be counted as an ace.

If you hold three aces, then the number of hands is

$$N_2 = 12 \cdot \binom{4}{3} \cdot \binom{5}{3} \cdot 44 = 21,120.$$

Here, the first term selects the rank of the other triple, and the second term counts the number of ways to choose those cards. The factor $\binom{5}{3}$ is the number of ways to choose three aces from a set including the bug as a potential fifth ace, and the last term fills in the odd card, which cannot be a card already chosen or any of the three remaining cards—possibly including the joker—that would elevate this hand to "four of a kind plus three of a kind".

The probability of two three-of-a-kinds is therefore

$$p = \frac{N_1 + N_2}{\binom{53}{7}} = \frac{68,640}{154,143,080} \approx 4.453 \times 10^{-4}.$$

At the Black Sheep Casino (now closed) in Cameron Park, California, pai gow poker was called *Pai Gow Poker Gold*. The hand rankings were extended to include Royal Flush + Royal Match: a 7-card hand consisting of a royal flush and a suited king and queen, without the joker. There are 4 natural royal flushes, and then 3 ways to select the suited king and queen, so the probability of this hand is

$$\frac{4 \cdot 3}{\binom{53}{7}} \approx 7.785 \times 10^{-8}.$$

In practice, a Royal Flush + Royal Match hand would be best played by breaking the royal flush and playing the kings in the back hand and the queens in the front hand: $KKAJT$ and QQ.

Pai Gow Express

The Commerce Casino offers a simpler version of PGP called *Pai Gow Express*, which deals 5 cards to each player from a 53-card deck including a fully wild joker. These hands must be divided into a 3-card back hand that outranks the 2-card front hand. 2-card hands are ranked as in standard PGP, but 3-card hands are ranked according to their relative scarcity as 3-card hands, which differs from the rankings of the corresponding 5-card hands. Table 7.27 shows the ranking of 3-card pai gow express hands. Two notable

TABLE 7.27: Pai Gow Express: 3-card hand rankings.

Three of a kind
Straight flush
Straight
Flush
Pair
High card

differences here are that 3 of a kind is the best possible hand and that a straight outranks a flush.

The house way of setting hands in pai gow express is much simpler than for 7-card pai gow poker. At a player's request, the dealer will set their hand for them, according to this house way:

1. Form a pair in the 2-card hand with a higher pair or complete hand (a hand higher than a pair) in the 3-card hand.

   Two-pair hands are played this way. The higher pair and the odd card go in the back, and the lower pair makes up the front hand.

2. Place an ace or king in the 2-card hand and at least a pair in the 3-card hand. The 3-card hand must be the best possible with an ace or king assigned to the 2-card hand.

   In the hand 2♣ T♣ K♠ 7♠ T♡, the 3-card hand would consist of a pair of 10s and the 7♣, and the 2-card hand would be K♠ 2♣, since the rule calls for the best possible 3-card hand rather than the best possible 2-card hand.

3. Make a complete 3-card hand in the back, with the highest possible 2-card hand.

   The hand T♢ 7♢ 3♡ 8♢ 4♠ contains a 3-card flush, which is placed as the 3-card hand. The 2-card hand is 4♠ 3♡. While this is the highest possible 2-card hand with a flush in the 3-card hand, it ranks very low.

   This hand configuration can nonetheless win both hands if the dealer holds a hand like 8♠ 6♠ 5♠ 3♢ 2♢, which would be set the same way: with the flush in the back hand. The player's flush beats the dealer's flush, and her 4-high front hand outranks the dealer's 3-high front hand.

4. Form a pair in the 3-card hand. This is a variant of rule #2 where there is no ace or king to assign to the 2-card hand.

5. In a case where only high-card hands are possible, assign the highest card to the 3-card hand and the second- and third-highest cards to the 2-card hand.

   When dealt 4♠ 5♢ 7♡ T♣ 9♡, the hand is divided into T♣ 5♢ 4♠ and 9♡ 7♡. 2-card straights and flushes have no standing in pai gow express, so the front hand is not scored as a flush, and the requirement that the back hand outrank the front hand is met.

Asia Poker

Deck composition:	53 cards
Hand size:	7 cards
Community cards:	None

Asia Poker is a 7-card table game in which players divide their cards into 3 separate hands. In this game, the 7 cards are split into a 1-card "Low" hand, a 2-card "Mid" hand, and a 4-card "High" hand. The cards can be divided in

$$\binom{7}{4} \cdot \binom{3}{2} \cdot \binom{1}{1} = 105$$

different ways; additional rules are in place to aid players in sifting through the possibilities. In addition to the names denoting the relative sizes of the hands, the three hands must be ranked in the order indicated: the Low hand must be equal to or lower in rank than the Mid hand, which in turn must be equal to or lower than the High hand. Players and dealer form their best hands, and the accounting is easy: any player winning at least 2 of 3 showdowns against the dealer's corresponding hands wins even money on their wager. The dealer takes all ties between comparable hands, which is a key source of the casino's advantage.

Asia Poker uses a 53-card deck, including a bug. Straights and flushes have no standing in the Mid hand: the only possibilities there are pairs and high-card hands.

Example 7.24. Suppose that a player is dealt the following cards:

$$A\diamondsuit\ 9\heartsuit\ 9\clubsuit\ 8\spadesuit\ 7\heartsuit\ 4\spadesuit\ 4\heartsuit$$

There are two pairs among the 7 cards, and the requirements for ranking the three hands suggest placing the 9s in the High hand and the 4s in the Mid hand. Making the ace the Low hand gives a strong hand which will win unless the dealer also has an ace in the Low hand, so one valid way to play these cards is

$$\begin{array}{rl} \text{High:} & 9\heartsuit\ 9\clubsuit\ 8\spadesuit\ 7\heartsuit \\ \text{Mid:} & 4\spadesuit\ 4\heartsuit \\ \text{Low:} & A\diamondsuit. \end{array}$$

Placing both pairs in the High hand leaves the Mid hand an 8-high hand, which is relatively weak even as two pairs in the High hand is a strong hand.

Suppose next that the dealer's cards are

$$Q\heartsuit\ Q\clubsuit\ T\diamondsuit\ 8\heartsuit\ 7\spadesuit\ 2\heartsuit\ 2\clubsuit.$$

The dealer is constrained by the house way of setting these cards. With two pairs, the dealer is required to split the pairs between the High and Mid hands (suggesting that this is a solid strategy for the player as well) unless holding a pair of aces or kings and no singleton jacks or queens. The dealer then sets the following hands:

$$\begin{array}{rl} \text{High:} & Q\heartsuit\ Q\clubsuit\ 8\heartsuit\ 7\spadesuit \\ \text{Mid:} & 2\heartsuit\ 2\clubsuit \\ \text{Low:} & 10\diamondsuit. \end{array}$$

In the showdown, the player's Mid and Low hands beat the dealer, while the dealer wins on the High hands. The player wins the round. ■

Asia Poker High hands consist of 4 cards, and so the hand ranking described in Table 7.28 is used to evaluate the High hand showdown. In particular, 4 of a kind beats a straight flush among 4-card hands.

TABLE 7.28: Asia Poker: 4-card hand distribution.

Hand	**Frequency**
Four of a kind	17
Straight flush	176
Three of a kind	2832
Flush	3828
Straight	4752
Two pairs	3096
One pair	6768
High card	11,580

Example 7.25. Consider 2-pair hands. When not holding the bug, there are

$$\binom{13}{2} \cdot \binom{4}{2}^2 = 2808$$

hands with 2 pairs. A 2-pair hand containing the bug must have the form $ABXX$, where B denotes the bug and X can be any rank other than aces. These hands are scarce, numbering only

$$4 \cdot 12 \cdot \binom{4}{2} = 288;$$

adding gives 3096 4-card hands with 2 pairs. ■

Table 7.28 notwithstanding, Asia Poker retains the standard order of hands where a flush and a straight both beat 2 pairs, although 3 of a kind rightly outranks both straights and flushes.

In practice, the house way for Asia Poker is designed to improve the dealer's Low hand within the rules and so to increase the chance of the dealer's Low hand beating or tying the players' hands. Scoring a win on the Low hands means that the dealer must win only one other hand to collect a player's wager, and players may be tempted to minimize the fact that all 3 hands are regarded equally in the three-part showdown.

Example 7.26. Suppose that the dealer's hand is

$$A\spadesuit\ A\heartsuit\ J\heartsuit\ 9\clubsuit\ 8\spadesuit\ 7\heartsuit\ 7\diamondsuit,$$

which contains a pair of aces and another low pair. The house way calls for the low pair to be set into the High hand and the Mid and Low hands set as ace-high hands, giving a High hand of 7♢ 7♡ 8♠ 9♣, a Mid hand of A♡ J♡, and a Low hand consisting of the A♠. This guarantees the dealer a win on the Low hand, which means that the only way the players can win is if both their Mid and Low hands beat the dealer. The dealer's ace-high Mid hand will be tough to beat. An alternate division would lead to a High hand of A♡ A♠ 8♠ 9♣, a Mid hand of 7♢ 7♡, and a Low hand of J♡. This assignment puts the Low hand at risk even as the other hands are stronger. ■

In Washington state, an optional Fortune Bonus bet is available to players. This side bet pays off based on the gambler's best 5-card hand, with some bonuses paid for especially high 7-card hands. Table 7.29 shows the payoffs.

TABLE 7.29: Asia Poker: Fortune Bonus bet pay table.

Player hand	**Payoff**
7-card straight flush	8000–1
Royal flush + Royal Match	2000–1
7-card straight flush with joker	1000–1
5 aces	400–1
Royal flush	150–1
Straight flush	50–1
4 of a kind	25–1
Full house	5–1
Flush	4–1
3 of a kind	3–1
Straight	2–1

On line 2 of Table 7.29, a Royal Match is a suited king and queen. A 7-card hand winning this 2000–1 payoff would look like ♣$AKQJT$ ♢KQ. There are exactly 12 ways to draw a royal flush with a Royal Match, making the chance of this hand approximately 1 in 11,148,713. Once again,we see a payoff that is far from a fair assessment of the difficulty of the underlying event—but of course, casinos don't stay long in business by offering fair games.

It is a curiosity that 7-card straight flushes are more numerous than royal flushes with a Royal Match (while still being extremely rare) at 32, but pay four times as much.

Silverado Stud Poker

Deck composition:	52 cards
Hand size:	3, 4, and 5 cards
Community cards:	2
Betting rounds:	2
Required bets:	5-card, 3-card
Optional bets:	4-card (Silverado Stud Plus)

Approved for play in South Dakota, *Silverado Stud Poker* deals each player 4 cards and uses 2 community cards. Players divide their hands into a 3-card hand and a 1-card hand, which are eventually combined with the community cards to form 3-card and 5-card hands. As with other carnival games, players play against a pay table rather than a dealer hand or each other.

Players make two equal bets to start, one each on the 5-card and 3-card hands. If the casino is using the Silverado Stud Plus variation, players have the option to make a third bet on the 4-card hand consisting of their hole cards. The cards are dealt, and players choose how to divide their cards into 3-card and 1-card subsets. There are two possible exceptions to this rule, both of which spare players the need to break up a good 4-card hand:

- A player dealt 4 of a kind may lay down her cards and push the 3-card hand bet, leaving action only on the 5-card bet. This qualifies the player for an increased 4 of a kind payoff on the 5-card hand.
- A player dealt 2 pairs has the same options. The 2-pair payoff on the 5-card hand is also increased when a player lays down 2 pairs.

Players dealt 4 of a kind or 2 pairs may choose not to divide their cards and play out both the 3-card and 5-card hands.

After the first community card is revealed, players have the option of raising their 5-card wager. The second community card is then turned up, and the bets are settled according to Tables 7.30–7.32 (page 251), which each show one of several state-approved pay tables.

Example 7.27. Three players receive the following hands and divide them as follows:

- Terry: $A\heartsuit$ $T\clubsuit$ 8$\spadesuit$ 8$\diamondsuit$.

  3-card hand: $T\clubsuit$ 8$\spadesuit$ 8$\diamondsuit$. 1-card hand: $A\heartsuit$.
- Dale: $T\spadesuit$ 6$\diamondsuit$ 3$\spadesuit$ 3$\clubsuit$.

  3-card hand: 6$\diamondsuit$ 3$\spadesuit$ 3$\clubsuit$. 1-card hand: $T\spadesuit$.
- Sandy: $K\diamondsuit$ $J\diamondsuit$ 6$\heartsuit$ 2$\diamondsuit$.

  3-card hand: $K\diamondsuit$ $J\diamondsuit$ 2$\diamondsuit$. 1-card hand: 6$\heartsuit$.

The community cards are the 9$\clubsuit$ and 3$\heartsuit$. The only paying hand is Dale's 5-card hand: 9$\clubsuit$ 6$\diamondsuit$ 3$\spadesuit$ 3$\heartsuit$ 3$\clubsuit$. Three of a kind pays 3–1. Terry and Dale both have 4-card hands containing a pair, which pays 1–1 if they made the Silverado Stud Plus bet. ■

Example 7.28. Table 7.31 contains a gap: the 4-card Silverado Stud Plus bet loses if the dealt 4-card hand is ace-high through 10-high. Find the probability of losing the bet.

Four-card ace-high hands number

$$4 \cdot \left[\binom{12}{3} - 2\right] \cdot (4^3 - 1) = 54,936.$$

TABLE 7.30: Silverado Stud Poker: 3-card hand pay table.

Hand	**Payoff**
Royal flush	30–1
Straight flush	15–1
3 of a kind	8–1
Straight	5–1
Flush	3–1
Pair	1–1

TABLE 7.31: Silverado Stud Poker: 4-card hand pay table.

Hand	**Payoff**
4 of a kind	200–1
Straight flush	50–1
3 of a kind	10–1
Flush	6–1
Straight	4–1
2 pairs	3–1
Pair	1–1
Less than 10 high	Push

TABLE 7.32: Silverado Stud Poker: 5-card hand pay table.

Hand	**Payoff**
Royal flush	500–1
Straight flush	100–1
Dealt 4 of a kind	40–1
Four of a kind	15–1
Full house	9–1
Flush	7–1
Straight	5–1
3 of a kind	3–1
Dealt 2 pairs	3–1
2 pairs	2–1
Pair, queens or better	1–1
Pair, 10s or jacks	Push

Here, the factor of 4 chooses the suit of the ace, $\binom{12}{3} - 2$ chooses the ranks of the 3 other cards, omitting the choices KQJ and 234 that lead to 4-card straights, and $4^3 - 1 = 63$ gives the number of ways to pick the suits of these 3 cards so that they don't all match the suit of the leading ace and create a flush.

Two modifications to the second factor are necessary to adapt this formula to king-high through 10-high hands:

- Reduce the 12 as needed to count all of the ranks below the high card, remembering that an ace dealt with a $K86$ produces an ace-high hand, not a king-high hand.
- Change the 2 to a 1, since only one 3-card combination can lead to a straight.

There are then a total of 161,028 four-card hands that are ace-high though 10-high, making the probability of a losing hand

$$\frac{161.028}{\binom{52}{4}} \approx .5948,$$

so nearly 60% of all 4-card hands will lose the Silverado Stud Plus bet. ■

7.4 Three Card Poker

Deck composition:	52 cards
Hand size:	3 cards
Community cards:	None
Betting rounds:	2
Required bets:	Ante
Optional bets:	Play, Pair Plus

Reducing the number of cards in a poker hand makes for a quicker game, and perhaps one that's easier to play. One common poker variant with fewer cards per hand is *Three Card Poker* (3CP), which has found a place in the table game lineup at a number of casinos.

The title gives the key to the game: players are dealt three cards apiece, and match their cards up against the dealer's hand—once again, this is a game where players face off against the dealer instead of each other. In 3CP, players make an Ante bet and then can choose, based on their cards, whether or not to fold or make a second Play bet. If the dealer fails to qualify by holding a hand that is at least queen-high, Ante bets pay even money and Play bets push. If the dealer qualifies, winning player hands pay even money on both the Ante and Play bets; if the dealer's hand is higher, then both bets lose. Additionally, the Ante bet pays a bonus, the Ante Bonus if the player's hand is a straight or higher, even if it is beaten by the dealer.

Example 7.29. What is the probability of the dealer qualifying?

There are $\binom{52}{3} = 22,100$ three-card hands, and $\binom{40}{3} = 9880$ of them contain no queens, kings, or aces. From this latter number, of course, we must remove any hand containing a pair or higher.

For example: Any card from 4 to J can be the highest card in a three-card straight flush; these hands run from 432 through $JT9$. There are 8 ranks and 4 suits to consider, for a total of 32 such hands.

These "exceptional" hands are counted in Table 7.33.

TABLE 7.33: Three Card Poker: Qualifying hands with no ace, king, or queen.

Hand	**Count**
Straight flush, $JT9$ or lower	32
Three of a kind, jacks or lower	40
Flush, jack-high or lower	$4 \cdot \binom{10}{3} - 32 = 448$
Straight, $JT9$ or lower	$32 \cdot 4 \cdot 4 - 32 = 480$
Pair, jacks or lower	$10 \cdot \binom{4}{2} \cdot 36 = 2160$

Adding up in this table gives 3160 possible qualifying hands with no card higher than a jack, and subtracting gives 6720 nonqualifying hands. The probability that the dealer qualifies is therefore

$$p = 1 - \frac{6720}{22,100} = \frac{15,380}{22,100} \approx .6959,$$

so the dealer qualifies about 70% of the time. This is much better for the player holding a good hand than the 56.32% chance of a Caribbean Stud Poker dealer holding a qualifying hand. ■

Armed with this information, we look next at the pay table for player hands that beat a qualifying dealer hand. The Ante Bonus pay table varies among casinos; a common payoff structure is 1 to 1 on straights, 4 to 1 on three of a kinds, and 5 to 1 on straight flushes [134]. The cost to the casino of the Ante Bonus is a function of the payoff values. If the bonus is x on straights, y on three of a kinds, and z on straight flushes, the Ante Bonus costs the casino

$$E(x,y,z) = x \cdot \frac{720}{22,100} + y \cdot \frac{52}{22,100} + z \cdot \frac{48}{22,100} = \frac{720x + 52y + 48z}{22,100}$$

cents per dollar wagered; there is no negative term since the Ante Bonus requires no additional bet. With the values listed above, this bet has a cost of about 5.29¢ per dollar. Some casinos recognize a "Mini Royal Flush": a suited AKQ, and reward it with a 50 to 1 Ante Bonus payoff. Adding this payoff to the pay table above raises the casino's cost less than a penny, to 6.10¢ per dollar.

There is no pay table for the Play bet—all bets are paid at 1 to 1—but there is an optional Pair Plus bet that pays off player hands of at least a pair at odds. This bet must be made with the Ante bet before the cards are dealt, of course. Pay tables can vary among casinos; one of the most common is Table 7.34. It should be noted that the Pair Plus bet is paid off even if the player's hand does not beat the dealer's or if the dealer fails to qualify.

TABLE 7.34: Three Card Poker payoffs: Pair Plus bet [134].

Hand	**Payoff**
Straight flush	40–1
Three of a kind	30–1
Straight	5–1
Flush	4–1
Pair	1–1

As we might expect, these payoffs are far below the probabilities of the various hands. In the case of a straight flush, there are 48 possible hands (any card except a deuce can be the high card in a three-card straight flush), and so

$$P(\text{Straight flush}) = \frac{48}{22,100} \approx \frac{1}{460}.$$

Strategy

What is the beacon hand for 3CP?

Once the player's 3-card hand is dealt, there are

$$\binom{49}{3} = 18,424$$

possible dealer hands to consider. In determining the beacon hand, it is necessary to compare a hypothetical player hand against each of these 18,424 hands. The lowest hand that returns over 100% draws the line between hands that should be played and hands where the better choice is to fold.

TABLE 7.35: Three Card Poker: Player results when holding an unsuited 678 hand [46].

Outcome	Number of hands
Win Ante and Play (+2 units)	12,626
Win Ante only (+1 unit)	5311
Lose (–2 units)	461
Tie (+0 units)	26

Example 7.30. If the player holds an unsuited 678, matching it against every possible dealer hand gives the results shown in Table 7.35.

The expected return from this hand, including the Ante Bonus payoff for a straight, is about \$3.61, making this straight worth the Play bet. ■

An additional consideration that guides our search for the beacon hand is that there is no point in making the Play bet on a hand that is not at least queen-high. It is impossible for a hand of $JT8$ or lower to beat a dealer qualifying hand, so making the Play bet on such a hand amounts to putting an additional unit at risk—and a fairly high risk, since it will be lost along with the Ante whenever the dealer qualifies—with no chance of collecting on that part of the combined Ante and Play bet [46].

Computer simulation leads to the answer: the minimum 3CP hand for which the expectation is greater than the $-\$1$ outcome achieved by folding and forfeiting the Ante bet turns out to be $Q64$: any player hand better than this should be backed with a Play bet and any lesser hand should be folded. There are 18,424 different dealer hands if the player holds $Q64$; Table 7.36 shows the possible outcomes if the $Q64$ hand consists of cards from 3 different suits.

TABLE 7.36: Three Card Poker: Player results when holding a 3-suited $Q64$ hand [46].

Outcome	Number of hands
Win Ante and Play (+2 units)	305
Win Ante only (+1 unit)	5758
Lose (–2 units)	12,335
Tie (+0 units)	26

Using these probabilities gives an expectation for the hand, if Ante and Play bets are both made, of

$$E = (2) \cdot \frac{305}{18,424} + (1) \cdot \frac{5758}{18,424} + (-2) \cdot \frac{12,335}{18,424} \approx -\$.993.$$

Since this value is greater than the –$1 outcome from folding, a $Q64$ hand justifies—though not by a lot—making a Play bet.

If that $Q64$ hand contains only 2 different suits, then one dealer hand moves from the Tie category to the Lose category; it's a second 3-card $Q64$ flush. This changes the expectation by −$.00011, not enough to shift the $Q64$ decision from play to fold.

On a $Q63$, the probabilities change just enough to tip the expectation to less than –$1, which favors folding any such hand regardless of whether it includes 2 or 3 different suits. Table 7.37 gives the hand frequencies.

TABLE 7.37: Three Card Poker: Player results when holding a 3-suited $Q63$ hand [46].

Outcome	**Number of hands**
Win Ante and Play (+2 units)	271
Win Ante only (+1 unit)	5747
Lose (–2 units)	12,380
Tie (+0 units)	26

The expectation is $E = -\$1.003 < -\$1$.

Following this one simple piece of advice limits the casino's advantage to a mere 3.37%—actually quite reasonable for a table game. If instead you choose to "mimic the dealer" by calling on every hand holding at least a queen, that gives the casino a 3.45% edge—not much of a change. The nonstrategy of always calling, on the other hand, raises the HA to 7.65%: more than double that when using the beacon hand strategy above [134].

As in CSP, this beacon hand presumes that players have no knowledge of the other players' cards, knowledge that might affect the dealer's chance of qualifying and suggest a different player strategy. If, for example, the other gamblers' hands are rich in queens, kings, and aces, the probability of the dealer qualifying falls, and so playing on a weak hand such as 975 might be justified in the hope of winning the Ante bet against a nonqualifying dealer hand. Note that it is not necessary that the player hand beat a nonqualifying dealer hand: if the dealer fails to qualify, then all players who have not folded receive even money on their Ante bets.

Conversely, if there are few high cards in the players' hands, making the Play bet even on a king-high hand might conceivably be a bad bet, since the chance of the dealer qualifying with a better hand increases. While there is no opportunity to raise one's bet in a favorable situation, as is the case with card counting in blackjack, any opportunity to reduce an already-low house advantage should be carefully considered, both by players and casino operators.

As with Caribbean stud poker during the Covid-19 pandemic in 2020, the state of Washington carefully considered the implications of dealing cards face up on the casino's advantage in 3CP [60]. In consultation with game design company Scientific Games, and under the assumption that only 3 players were permitted at the table at any one time, the following results were found.

- Limiting the table to 3 players diminished the effect of revealing cards.
- Consider the case where a player holds a jack-high or lower hand. In general, if the probability of the dealer qualifying is p, the expected value of the hand if the Play bet is made is

  $$E = (-2) \cdot p + (1) \cdot (1 - p) = 1 - 3p,$$

  which is only greater than −1, favoring a Play bet, if $p < \frac{2}{3}$. On a $JT8$, the highest possible jack-high hand, this probability is

  $$p = \frac{13,147}{18,424} \approx .7135 > \frac{2}{3}$$

  and the expectation is −\$1.14, provided that we know nothing else about the cards held by other players [46].

  If 4 or more of the 6 exposed cards in other players' hands are queens, kings, or aces, this becomes a marginally playable hand. If 5 or 6 cards are this high, then any jack-high or lower hand should be played. The probability of k out of 6 cards being queens or higher is

  $$\frac{\binom{12}{k} \cdot \binom{40}{6-k}}{\binom{52}{6}}.$$

  If we sum this formula for $k = 4, 5$, and 6, we get .0206, so there is just slightly more than a 2% chance of this small advantage being activated. Of course, if the player holds at least $Q64$, this increased advantage is no advantage at all, since the information does not change their playing strategy.
- If all 6 cards held by the other players are low cards (everything less than a queen), an event with probability

  $$\frac{\binom{40}{6}}{\binom{52}{6}} \approx .1885,$$

  then some of the lowest playable hands—$Q6x$ and $Q7y$—should be folded because the chance of the dealer qualifying, and the expected strength of that qualifying hand, have risen. Hands ranking at or above $Q8x$ should be played regardless of the exposed cards. Once again, the opportunity to seize an advantage based on the exposed cards is very limited.

- A round of simulation suggested that if all 6 cards were exposed, the net effect on the house edge would be 0.2% at most, assuming perfect play on the part of the gamblers.

With the risk deemed to be low, Washington casinos, at Scientific Games' request, were permitted to deal games such as 3CP face up.

Six Card Bonus

At the Bay 101 Casino in San Jose, California, Three Card Poker includes an optional Six Card Bonus bet, which is based on the best 5-card hand that can be made from the 6 cards in the player's and player/dealer's hands. Table 7.38 shows the payoff odds on this bet.

TABLE 7.38: Three Card Poker: Six Card Bonus pay table.

Royal flush	1000–1
Straight flush	200–1
Four of a kind	100–1
Full house	20–1
Flush	15–1
Straight	10–1
Three of a kind	7–1

The probability of a 5-card royal flush in 6 cards is

$$\frac{4 \cdot 47}{\binom{52}{6}} = \frac{1}{108,290},$$

exactly 6 times the probability of a royal flush in 5 cards.

Three Card Prime

Three Card Prime is a variation on Three Card Poker that adds two optional prop bets. The game was on offer at the Santa Fe Station Casino in Las Vegas in 2018. The *Prime* bet pays off at 3–1 if the player's three cards are all the same color, and 4–1 if the dealer's hand also holds 3 cards of that same color, so the payoff increases if all 6 cards are the same color. This bet, if made, has action even if the player's 3-card poker hand loses to the dealer's or if the dealer fails to qualify with at least a queen.

The probability that the player holds a 3-card hand with all cards the same color is

$$2 \cdot \frac{\binom{26}{3}}{\binom{52}{3}} = \frac{4}{17},$$

where the factor of 2 accounts for the choice of color. If the player's hand has 3 cards of one color, the probability that the dealer's hand contains 3 cards of that color is

$$\frac{\binom{23}{3}}{\binom{49}{3}} = \frac{253}{2632} \approx .0961,$$

a bit less than 10%. It follows that the expected value of a \$1 Prime bet is

$$E = \frac{4}{17}\left[(3)\cdot\frac{2379}{2632} + (4)\cdot\frac{253}{2632}\right] + (-1)\cdot\frac{13}{17} \approx -\$.0362,$$

which results in a very reasonable 3.62% HA.

A simple *Pair* wager at Three Card Prime pays off if the player's hand—whether it beats the dealer or not—is at least a pair. The payoffs increase if a fourth card dealt to the table, independent of either hand, is a 2; this is called the "Loose Deuce". Table 7.39 shows the payoff odds on the Pair bet.

TABLE 7.39: Three Card Prime: Pair wager pay table.

Hand	**Payoff**	**Loose Deuce Payoff**
Mini royal	50–1	75–1
Straight flush	40–1	50–1
Three of a kind	30–1	40–1
Straight	6–1	8–1
Flush	3–1	4–1
Pair	1–1	2–1

Example 7.31. There are 44 three-card straight flushes in a standard deck. Eight of these contain a 2; for those, there are 3 deuces remaining that can trigger the higher 50–1 payoff. This gives 24 "enhanced" straight flushes. Each of the remaining 36 straight flushes may be combined with any of the 4 deuces, generating an additional 144 enhanced straight flushes, for a total of 168.

Continuing the count, the 8 straight flushes with a deuce can be combined with any one of the 46 non-deuces, and the 36 straight flushes without a deuce can draw 45 different fourth cards, for a total of $8 \cdot 46 + 36 \cdot 45 = 1988$ straight flushes whose fourth card is not a deuce.

The player's hand plus the bonus card can be dealt in

$$\binom{52}{3} \cdot 49 = 1,082,900$$

ways, so the probability of a straight flush is

$$\frac{2154}{1,082,900} \approx .0020.$$

The straight flushes contribute

$$(40) \cdot \frac{1988}{1,082,900} + (50) \cdot \frac{168}{1,082,900} \approx \$.0812,$$

just over 8¢, to the expected return of a $1 Pair bet. ■

Three-Card Lowball Poker

Three-Card Lowball with Triple Draw (page 95) is a poker game where players compete against each other, and is suggested as an option for home dealer's choice games. Several carnival games are also called *Three-Card Lowball Poker*. All deal 3-card hands, and players compete against the dealer to assemble the lowest hand with 3-card straights and flushes included as high hands. Aces always count low, so QKA is a king-high hand, not a straight. The 22,100 possible 3-card hands are distributed according to Table 7.40.

TABLE 7.40: Three-Card Lowball Poker: Hand count.

Hand	Number of hands
Straight flush	44
Three of a kind	52
Straight	660
Flush	1100
Pair	3744
High card	16,500

Note here that straights are less common than flushes—but since players are going for low, this means that a flush still beats a straight.

Since there are 16,500 hands which consist of 3 unsuited cards of different ranks and not in sequence, getting one of those hands on the deal is common. The chance to improve it with one or two draws, despite the risk of pairing a card or drawing a high card as replacement, is significant. Each version of three-card lowball poker allows players to discard some of their cards and draw new ones in an effort to get a lower hand. The games differ in the number of draws permitted to players as they seek to improve their hands and in how high-card hands are scored and ranked.

Single Draw

The simplest version of 3-card lowball poker, developed by Heather Ferris of Vegas Aces Services, permits only a single draw. Players begin by making equal Ante and Bonus bets. Once the cards are dealt, they may discard up to 2 of their cards and draw replacements. The dealer plays her hand according to the house way, which is to discard any card that's a 10 or higher, as well as to break up straights or flushes. All completed dealer hands move forward to the showdown; there is no question of the dealer qualifying or not.

Winning hands pay 1–1 on the Ante. Bonus bets are paid according to Table 7.41, provided that the player's hand beats the dealer's.

TABLE 7.41: 3 Card Lowball Poker (1 draw version) Bonus bet pay table: Player's hand must beat the dealer's [31].

Hand	**Payoff**
4–2–A, one color	10–1
4–2–A, two colors	6–1
4–3–A	5–1
5 high	3–1
6 high	2–1
7 high	1–1
Other hands	Push

The probability of receiving an unsuited $42A$ hand on the deal is

$$\frac{4^3 - 4}{\binom{52}{3}} = \frac{60}{22,100}.$$

Of these, 12, or 20%, are single-color hands. Of course, in the unlikely event that the dealer also holds $42A$, the Bonus and Ante bets both push, even with so fine a hand.

Double Draw

A product of Total Gaming Science, the 2-draw version of 3-card lowball poker permits players and dealer to draw up to 2 cards on the first draw and one card on the second draw. This game also changes how high-card hands are valued. These hands are ranked according to the sum of their cards' ranks, with aces counting 1, jacks 11, queens 12, and kings 13. Since this is a lowball game, the lower hand wins when player and dealer hands are compared. This leads to different ordering of some high-card hands.

Example 7.32. In the one-draw version of three-card lowball, the hand $T♣$ $8♡$ $6◇$ would rank lower than $Q◇$ $4♣$ $3♠$, as a 10-high hand beats a queen-high hand, and none of the other cards affect the outcome. In the two-draw game, every card contributes to the hand's value. A $T86$ hand scores 24 points, higher than the 19 points of $Q43$, and so $Q43$ wins. ■

Example 7.33. The player is dealt $A♠$ $A♡$ $7◇$, and discards an ace on the first hand, drawing the $3♡$ for a hand totaling 11. He opts not to discard in the second round, and so stands with 11.

The dealer's hand is revealed: $Q\spadesuit$ $J\spadesuit$ $4\heartsuit$. The house way for this game calls for the dealer to break up pat hands of a pair or higher and to discard all cards 8 or higher, so she discards the face cards. Her new hand is $9\spadesuit$ $9\diamondsuit$ $4\heartsuit$, and the house way allows her only to discard one of the two 9s. Her final hand is $J\diamondsuit$ $9\diamondsuit$ $4\heartsuit$ for a total of 24, which loses to the player. ■

The Bonus bet in this game pays off on winning low hands, following Table 7.42.

TABLE 7.42: 3 Card Lowball Poker (2 draw version) Bonus bet pay table: Player's hand must beat the dealer's [149].

Hand value	**Payoff**
7 points	5–1
8 points	4–1
9 points	3–1
10 points	1–1
Other hands	Push

This Bonus bet appears less lucrative than the corresponding bet in the one-draw game. The only 7-point hands are $42A$, which pay better with only 1 draw, but are perhaps easier to draw into with a second chance to exchange a card.

7.5 Four Card Poker

Deck composition:	52 cards
Hand size:	5 cards dealt, 4 used
Community cards:	None
Betting rounds:	2
Required bets:	Ante
Optional bets:	Play, Aces Up

Four Card Poker (4CP) pits the gambler against the dealer in a head-to-head game matching four-card poker hands. In addition to the usual practice of paying off winning hands at less than true odds, as was seen in CSP, the casino derives an additional advantage from the rule that the player gets 5 cards to construct a 4-card poker hand, while the dealer gets 6. One of the dealer's cards is dealt face up as an aid to players in judging the hand's strength. There is no dealer qualifying hand in 4CP; all dealer hands face off against players who make the optional Play bet, regardless of the hand's strength or weakness.

Betting in 4CP also occurs in several stages:

- The player makes an initial Ante bet before the cards are dealt.
- Upon seeing his or her cards and forming the best possible four-card hand, the player may fold, forfeiting the Ante bet, or may make an additional Play bet ranging from one to three times the Ante bet.
- An optional Aces Up bonus bet, made before the deal, pays off in accordance with Table 7.43 if the player's hand is a pair of aces or higher, regardless of the dealer's hand.

TABLE 7.43: Four-Card Poker: Aces Up bonus bet pay table.

Hand	**Payoff odds**
Four of a kind	50–1
Straight flush	40–1
Three of a kind	8–1
Flush	5–1
Straight	4–1
Two pairs	3–1
Pair of aces	1–1

The player and dealer hands are compared, and the higher hand wins with the player taking ties. The payoff is even money on both the Ante and Play bets. At the Soaring Eagle Casino, an automatic bonus is paid on the Ante bet if the player's hand is three of a kind or higher, even if the hand is beaten by the dealer. Table 7.44 contains the payoffs.

TABLE 7.44: Four-Card Poker: Automatic Bonus pay table.

Four of a kind	25–1
Straight flush	20–1
Three of a kind	2–1

Example 7.34. Tables 7.43 and 7.44 indicate that four of a kind is better than a straight flush in 4CP, which is the opposite order from five-card poker, as we saw in Section 1.4. Careful counting will confirm this.

There are 13 possible four of a kinds, one for each rank. Any one of them may be dealt with any of the remaining 48 cards as the fifth card. This gives a total of $13 \cdot 48 = 624$ ways to draw four of a kind. This is exactly the number of four-of-a-kind hands in standard five-card poker.

Four-card straight flushes are more common than their five-card counterparts. As with five-card poker, we consider the cards comprising the flush in order. Note that the royal flush is not broken out as a separate hand, so any card from ace through jack may be the lowest card in a four-card straight flush, for a total of 44 distinct straight flushes. As with four of a kinds, the fifth card in the player's hand may be any of the remaining 48 *except* the one card that completes a higher straight flush.

For example, consider the straight flush 5678♣. If the fifth card dealt to the player is the 9♣, then the four-card hand will be played as 6789♣. All straight flushes except those starting with a jack (a total of 40) require that this deletion occur. We have

- 40 four-card straight flushes whose lowest card is ace through 10. Each can be combined with 47 other cards in the deck.
- 4 four-card straight flushes consisting of a suited JQKA. These can be filled out to a 5-card hand by any of the other 48 cards.

This gives

$$40 \cdot 47 + 4 \cdot 48 = 44 \cdot 48 - 40 = 2072$$

four-card straight flushes, over three times as many straight flushes as four of a kinds. ■

These results are for the player's hand. For the dealer's hand, we must count the number of possible four-card straight flushes and four-of-a-kinds when six cards are dealt. There are 14,464 four-of-a-kinds and 47,752 straight flushes, confirming the hand ranks used in 4CP.

As with 3CP, 4CP comes with a major strategy decision: whether or not to make the Play bet based on one's cards. Unlike CSP, there is no question of the dealer qualifying or not, so the only question to be answered is whether or not your hand has a good chance of beating the dealer's.

The computer simulation approach used for 3CP is not viable with current technology due to the sheer number of hands to be considered. In 3CP, there were $22,100 \cdot 18,424 = 407,170,400$ different player vs. dealer comparisons to make; this is easily done by considering all cases. In 4CP, however, the number of possible player vs. dealer combinations is

$$\binom{52}{5} \cdot \binom{47}{6} = 2,598,960 \cdot 10,737,573 = 27,906,522,724,080$$

—over 27 trillion matchups; about 68,500 times greater. This is simply too many for efficient calculation [45].

The Play bet pays off at even money, so you should make that bet if your hand falls into the top half of all four-card poker hands. The rack card (a card available at the table that explains the rules, and possibly strategies, for a game) at the Soaring Eagle Casino states that you should make the Play bet if your hand is a pair of 3s or higher. Is this sound advice? What is the beacon hand that separates hands worth a Play bet from those which should be folded?

Since the player is dealt five cards to make his or her best four-card hand, we shall consider five-card combinations, of which there are 2,598,960. How many of these hands lack even a pair when reduced to their best four cards? As with five-card hands, the standard procedure is to compute the number of higher hands and subtract that total from this number. The hand distribution for 4CP hands is given in Table 7.45.

TABLE 7.45: Four Card Poker: Hand distribution.

Hand	**Frequency**
Four of a kind	624
Straight flush	2072
Three of a kind	58,656
Flush	114,616
Straight	101,808
Two pairs	123,552
One pair	1,098,240
High card	1,302,540

Our beacon hand would be hand #1,299,480, which is still among the list of high-card hands. While the Soaring Eagle's advice is valid, there are a number of hands—pairs of 2s as well as some high-card hands—that should be played rather than folded. Of course, if your high-card hand cannot beat the dealer's upcard—as when your hand is king-high and the dealer shows an ace, for example—you should fold immediately. This is one situation where the value of the dealer's upcard is useful as a guide to gameplay.

The apparently important question of how big a Play bet to make is, in practice, quite simple to answer. When you make the Play bet, you are doing so because you expect that your hand will win. If this bet is worth making, it is worth making for the maximum possible value: three times your Ante bet. Any smaller Play bet is leaving expected winnings on the table.

Crazy 4 Poker

Deck composition:	52 cards
Hand size:	5 cards
Community cards:	None
Betting rounds:	2
Required bets:	Ante, Super Bonus
Optional bets:	Play, Queens Up

Crazy 4 Poker is a table game where players and dealer are dealt 5 cards each and challenged to make the best possible 4-card poker hand from them.

Straights, flushes, and straight flushes all consist of 4 cards; royal flushes are not broken out as a separate type of hand.

As in Three-Card Poker, there are two rounds of betting: an Ante and Super Bonus wager are made before the cards are dealt. Players then receive 5 cards each, and may then make a Play bet and face off against the dealer's best 4 cards if their hands are sufficiently strong. The Play bet is the same amount as the Ante bet unless the player holds a pair of aces or higher, in which case the Play bet may be triple the Ante bet. If the dealer's hand is not at least king-high, it fails to qualify, and players are paid even money on their Play bets while Ante bets push. Should the dealer qualify, a player whose hand outranks the dealer's wins even money on both Ante and Play bets.

The required Super Bonus bet pays off in accordance with Table 7.46. This

TABLE 7.46: Crazy 4 Poker: Super Bonus bet pay table.

Hand	**Payoff**
Four aces	200–1
4 of a kind	30–1
Straight flush	15–1
3 of a kind	2–1
Flush	3–2
Straight	1–1

bet pays off on a straight or higher even if the player loses to the dealer, but if the player beats the dealer with a hand less than a straight, the Super Bonus bet pushes. If the dealer fails to qualify, the Super Bonus bet still has action.

Example 7.35. In a three-handed game, the following cards are dealt:

- Robin: $Q\clubsuit$ $9\heartsuit$ $8\heartsuit$ $3\diamondsuit$ $2\spadesuit$. The $2\spadesuit$ is discarded, and this hand is queen-high.
- Dale: $A\clubsuit$ $K\clubsuit$ $K\spadesuit$ $Q\diamondsuit$ $T\diamondsuit$. This hand plays as a pair of kings.
- Pat: $Q\spadesuit$ $8\diamondsuit$ $5\spadesuit$ $4\spadesuit$ $4\clubsuit$. On discarding the $5\spadesuit$, this becomes a pair of 4s.

Robin folds, while Dale and Pat make a $1\times$ Play bet. The dealer's hand is then turned up: $Q\heartsuit$ $J\diamondsuit$ $9\clubsuit$ $6\clubsuit$ $3\spadesuit$: only queen-high. Since the dealer fails to qualify, Dale and Pat win on their Play bets. Their Super Bonus bets push, since both players' pairs beat the dealer but rank lower than a straight. Robin's Super Bonus bet remains in play, but the hand loses. ■

Recalling our work in Chapter 5, we are not surprised when 4-card hands have a different order than 5-card hands. There are $13 \cdot 48 = 624$ four-of-a-kind hands in Crazy 4 Poker: any of the 13 quadruples may be combined with any other card in the deck, which is discarded in reducing the hand from 5 cards to 4.

Straight flushes number 2072: There are

$$(4 \cdot 11) \cdot 48 = 2112$$

5-card hands containing at least one 4-card straight flush. From this number, we subtract the 40 five-card straight and royal flushes, since we have counted them twice in this accounting.

Some Crazy 4 Poker tables offer additional side bets. The *Queens Up* bet pays off on any player hand of a pair of queens or higher. Table 7.47 shows the payouts.

TABLE 7.47: Crazy 4 Poker: Queens Up side bet pay table.

Hand	**Payoff**
4 of a kind	50–1
Straight flush	40–1
3 of a kind	7–1
Flush	4–1
Straight	3–1
2 pairs	2–1
Pair, queens or better	1–1

We have computed the number of 4-card four-of-a-kind hands and straight flushes. The chance of 4 of a kind paying 50–1 is 1 in 4165; the probability of a straight flush with its 40–1 payoff is approximately 1 in 1254. Counting the remaining paying hands of QQ or higher gives a HA for Queens Up of 6.78%. If the payoff on 3 of a kind is raised to 8–1, as is the case in a Queens Up pay table approved for use in Washington state casinos, the HA drops to 4.52%.

7.6 California Games

Commercial non-tribal card rooms in California are subject to State Penal Code 330, passed in 1873 and amended in 1891, which outlaws house-banked games throughout the state [152]. House-banked games are games where gamblers compete against the casino, such as craps, roulette, baccarat, and blackjack. Penal Code 330 restricts commercial casinos to card games where players compete directly against each other; this encompasses a variety of poker-based table games developed specifically for the California market.

Since the card room is merely hosting the game, it has no stake in the action and so cannot count on a house advantage to make a profit, nor is there a central pot for each dealt hand from which the card room may collect a rake. Card rooms make money by charging a collection fee on every hand. This fee is set by the card room; this may be a flat fee per player per hand or a percentage, often 1%, of a player's wager with a minimum of $1.

Some California card rooms offer modified versions of poker that conform to the law while still retaining much of the game's essence. Because players must compete against each other and not the house, there are provisions for a player/banker, who functions as the dealer, faces off against the other players, and must pay all winning hands even as he or she collects from all losing players. A casino employee deals all of the cards; the player/banker designation is in name only. As each player competes only against the player/banker and not the other players at the table, these games are properly classified as carnival games. The player/dealer is sometimes playing the positive advantage side of the game, and so card rooms charge a fee to this player as well.

Pokara

Deck composition:	60–400 cards
Hand size:	3 cards
Community cards:	0
Betting rounds:	1
Required bets:	Ante
Optional bets:	Play, Bonus

The Gardens Casino offers a three-card draw poker game called *Pokara*. Pokara is played with 3–20 decks of 20 cards each: the tens through aces from all 4 suits as used in royal hold'em. For optimal gameplay, the casino recommends that 20 decks, for a total of 400 cards, be used.

A hand of Pokara is played with each player dealt an initial 3-card hand face up. The player/dealer receives a 3-card hand with 2 cards dealt face down. Remembering that the goal is to beat the player/dealer's hand, players have the option to stand on their initial hand or discard and replace one card. A player who discards may double their wager before drawing, but risks losing the hand at once if the draw does not improve their hand.

Example 7.36. The lowest initial hand is an unsuited KJT. A one-card draw, discarding the 10, can only improve the hand unless another 10 is drawn, either by pairing the king or jack or by replacing the 10 with a queen or ace. If the king and jack are suited, a 10 of the same suit also improves the initial hand. ■

Once all players have made their choice, the player/dealer turns over her hole cards. She is compelled to follow the house way when deciding whether or not to draw. The house way is shown in Table 7.48, although if the 3 cards form a flush, the player/dealer stands rather than drawing.

TABLE 7.48: Pokara: House way for the player/dealer hand. Flushes discard no cards.

Dealt hand	**Discard**
AKJ	J
AKT	T
AQJ	A
AQT	T
AJT	T
KQT	T
KJT	T
AAT	T
KKT	T
QQT	T
JJT	T
TTJ	J

Line 3 of Table 7.48 is the only one where the house does not discard the lowest card. This is because discarding the ace increases the chance of completing a 3-card straight with a king or 10.

Any player/dealer hand not listed in Table 7.48 must stand as dealt. As with the players, if the draw does not improve the player/dealer's hand, it loses to all player hands that have not already lost. As in blackjack, the player/dealer's edge comes from the fact that the players must complete their hands first. Winning bets pay even money.

Example 7.37. Often, a casino game's house way is also the best strategy for players. A player holding $K\heartsuit\ K\spadesuit\ Q\clubsuit$ would be advised to stand without discarding, regardless of the dealer's upcard. The only way to improve this hand by discarding the queen would be to draw a king or ace. Roughly 60% of the time, for the exact probability depends on the cards that have been dealt in previous hands and not replaced, the hand will fail to improve and will lose before the dealer's hand is completed. ■

With so much duplication of cards, the highest 3-card Pokara hand is a *trip flush*: 3 cards of the same rank and suit. The probability $P_T(n)$ of a trip flush, expressed as a function of the number of decks n dealt from the top, is

$$P_T(n) = \frac{20 \cdot \binom{n}{3}}{\binom{20n}{3}} = \frac{(n-1)(n-2)}{(20n-1)(20n-2)}.$$

$P_T(n)$ increases as n increases, reaching a maximum of $\frac{3}{1393} \approx .00215$ with 20 decks.

By contrast, the probability of a 3-card royal flush (AKQ) is

$$P_R(n) = \frac{4n^3}{\binom{20n}{3}} = \frac{24n^3}{20n(20n-1)(20n-2)}.$$

We have $P_R(n) > P_T(n)$ for all integer values of n between 3 and 20, so a trip flush correctly outranks a royal flush.

Table 7.49 shows the rank of Pokara hands.

TABLE 7.49: Pokara hand ranking.

Trip flush
Royal flush
Straight flush
Three of a kind
Flush
Straight
Pair
High card

Assume that 20 decks are in use. There are then

$$\binom{400}{3} = 10,586,800$$

possible 3-card hands from a fresh shoe. The median hand, ranking 5,293,400 on the list, is QQJ unsuited.

Example 7.38. Note that suited pairs hold no special status in Pokara. What is the probability of a suited pair?

There are $20 \cdot \binom{n}{2}$ ways to choose a suited pair from a full n-deck shoe. There are then $12n$ choices for the third card, which cannot be of the same rank or suit as the pair. This ensures that the hand is a suited pair and not three of a kind or a flush. We have

$$P(\text{Suited pair}) = \frac{20\binom{n}{2} \cdot 12n}{\binom{20n}{3}} = \frac{18n(n-1)}{(20n-1)(10n-1)}.$$

This probability is lowest for $n = 3$, when it's .0631. With 20 decks, the probability rises to its maximum: .0862.

In the limit as the number of decks approaches ∞, the probability of a suited pair approaches .0900. ■

TABLE 7.50: Pokara Bonus bet pay table.

Initial hand	**Payoff**
Trip flush in clubs	200–1
Trip flush, other suits	50–1
Royal flush	30–1
Straight flush	20–1
Three of a kind	8–1
Flush	3–1

Pokara also offers players an optional bonus bet, which is based on the rank of their initial 3-card hand. Table 7.50 shows the pay table.

The probability of a trip flush in clubs, with n decks in use, is

$$P(n) = \frac{5 \cdot \binom{n}{3}}{\binom{20n}{3}} = \frac{(n-1)(n-2)}{4(20n-1)(20n-2)}.$$

If $n = 20$, this probability reaches its maximum: $P(20) \approx 5.384 \times 10^{-4}$, and the chance of a trip flush in any other suit with 20 decks is simply 3 times this, approximately .00162.

The HA of the bonus bet varies with the number of decks, which is expected, since the probability of drawing a particular winning hand increases with n while the payoffs remain fixed. If $n = 3$, the house advantage is 26.13%; it drops to 5.94% with 20 decks in use.

These numbers are calculated under the assumption that a fresh shoe is in use; as cards are dealt from the shoe, the HA will fluctuate, although it is highly unlikely that the shoe will be depleted so far that the players have an edge. Consider the very extreme case where Pokara is dealt from a 20-deck shoe from which all of the hearts have been removed. The probability of a trip flush in clubs from this 300-card deck is

$$p = \frac{5 \cdot \binom{20}{3}}{\binom{300}{3}} \approx .0013,$$

approximately 1 chance in 782. This hand then contributes $.0013 \cdot 200 \approx$ \$0.2559 to the expected value of the Bonus bet.

This value of p is also the probability of a trip flush in spades or diamonds, which add $50p$ each to the expectation. Continuing through the other paying ranks gives $P(\text{Royal flush}) \approx .0054, P(\text{Straight flush}) \approx .0162$, and so on. Ultimately, the expectation is indeed positive—but the probability of all 100 hearts rising to the top of a freshly-shuffled shoe is

$$\frac{\frac{100!}{(20!)^5} \cdot \frac{300!}{(20!)^{15}}}{\frac{400!}{(20!)^{20}}} = \frac{100! \cdot 300!}{400!} \approx 4.4606 \times 10^{-97},$$

where we use the formula for distinguishable permutations found on page 17. While this may be an exceptionally unlikely case, the low probability illustrates the extreme unlikelihood of a Pokara shoe ever entering a state where the Bonus bet has a player edge.

Supreme 99

Deck composition:	52 cards
Hand size:	4 cards
Community cards:	0
Betting rounds:	1
Required bets:	Play
Optional bets:	Pair Fortunes

Supreme 99 is a division game: a combination of poker and baccarat that uses 2-card hands. The game has been offered at the Seven Mile Casino in Chula Vista. The player/banker faces off against each player individually. While baccarat is frequently dealt from a shoe containing 6 or 8 decks of cards, Supreme 99 is dealt from a single deck, reshuffled after every hand. Unlike other carnival games based on poker, Supreme 99 does not follow an Ante/Play bet structure.

Players are dealt 4 cards, which they must arrange into two 2-card hands: High and Low. The High hand must outrank the Low hand. The highest-ranking hand is a pair, with aces high and deuces low. Hands without a pair are not read as high-card hands, but are ranked using their baccarat score: the ranks of the 2 cards are added with face cards counting 0. Any total over 9 drops the tens digit, so a hand like 6♠ 5♡ would have a value of 1—but it would beat a KQ hand, 1–0. The highest possible non-pair hand has a value of 9.

The player/banker must arrange her cards according to the house way:

- If dealt two pairs, the higher pair is set in the High hand.
- If dealt one pair, the pair goes in the High hand.
- For a no-pair hand, the Low hand is maximized without fouling the two hands. The players' hands cannot be fouled, but the player/banker's hand is foul if the Low hand outranks the High hand.

Example 7.39. The hand 6♣ J♠ 8♢ 7♢ can be legally set with the 2 hands scoring 6 and 5, 7 and 4, or 8 and 3. The house way would set the hands as 6 and 5: J♠ 6♣ and 8♢ 7♢. ■

In the showdown, the player wins when both hands beat the player/banker's corresponding hands. Winning bets pay even money. If one player hand wins and the other loses, then the hand is a push. The player/banker wins all "copy hands": hands that are identical in rank. However, if both hands copy, the player wins a 4–1 payoff on his original bet.

Supreme 99 players may choose an optional Pair Fortunes bonus bet that is based on their initial 4-card hand. This bet pays off if the High or Low hand contains a pair, in accordance with Table 7.51.

TABLE 7.51: Supreme 99: Pair Fortunes bonus bet pay table.

Player's hand	**Payoff odds**
4 of a kind	300–1
Two pairs	15–1
Pair of 9s	6–1
Any other pair	1–1

The probability of 4 of a kind is simply

$$\frac{13}{\binom{52}{4}} = \frac{13}{270,725} = \frac{1}{20,825},$$

and so the 300–1 payoff wildly undervalues this rare hand. The probabilities of the other 3 winning hands are

$$P(\text{2 pairs}) = \frac{\binom{13}{2}\binom{4}{2}^2}{\binom{52}{4}} \approx .0104.$$

$$P(\text{Pair of 9s}) = \frac{\binom{4}{2}\binom{12}{2}\cdot 4^2}{\binom{52}{4}} \approx .0234.$$

$$P(\text{Any other pair}) = \frac{12\cdot\binom{4}{2}\cdot\binom{11}{2}\cdot 4^2}{\binom{52}{4}} \approx .2340.$$

The expected value of a \$1 Pair Fortunes bet is −18.77¢, giving the player/banker an 18.77% edge. Supreme 99 charges no collection fee to players; only the player/banker is required to pay a per-hand fee. This fee is based on the total action at the table, including both the main bet and any Pair Fortunes bets made.

Triple Action Poker

Deck composition:	28 cards
Hand size:	5 cards
Community cards:	3
Betting rounds:	1
Required bets:	Ante
Optional bets:	Play, Bonus, Flop

At the Bicycle Casino, *Triple Action Poker* pits gamblers against a designated player/dealer in a table game with some similarities to Texas hold'em. Triple Action Poker uses a 28-card deck with 8s through aces in 4 suits. Play begins with an Ante bet and a collection fee paid by each player; optional Bonus and Flop bets of the same amount as the Ante may also be made. Each player is dealt 2 hole cards face down, and the player/dealer receives one card face up and one face down. Based on the 3 cards they can see, each player either folds, forfeiting their Ante, or makes an additional Play bet of the same size as the Ante bet. A flop of 3 community cards is then dealt, and the dealer's hole card is exposed. If the player/dealer's hand is at least a pair of 9s, he or she qualifies and the 5-card hands face off, with the higher hand winning. Ante and Play bets pay even money when the player wins.

In another instance of a common rule, if the player/dealer's hand fails to qualify, then all players who did not fold win even money on their Ante bet while their Play bet pushes. Additionally, player hands of a full house or higher receive an Ante Bonus payoff on their Ante bet whether or not they beat the player/dealer. The amount of the bonus is shown in Table 7.52.

TABLE 7.52: Triple Action Poker: Ante Bonus pay table.

Player's hand	**Payoff odds**
Royal flush	50–1
Straight flush	20–1
Flush	6–1
Four of a kind	4–1
Full house	2–1

Table 7.52 shows the correct order of poker hands drawn from the depleted deck used in Triple Action Poker. Flushes outrank four of a kinds in this game, as direct calculation shows. There are 4 royal flushes and 8 straight flushes in

the 28-card deck. Aces rank high in Triple Action Poker, so A89TJ is not a straight. The total number of flushes, with these 12 hands removed from the count, is

$$4 \cdot \binom{7}{5} - 12 = 72.$$

The number of four of a kinds is

$$7 \cdot 24 = 168,$$

over twice as great as the number of flushes.

The Ante Bonus payoff requires no additional wager; its expected value is about 3½¢. Players who want to make an additional bet in the hope of a good 5-card hand may choose to match their Ante bet with a Bonus prop bet. The Bonus bet pays off according to Table 7.53.

TABLE 7.53: Triple Action Poker: Bonus bet pay table.

Player's hand	**Payoff odds**
Royal flush	100–1
Straight flush	50–1
Flush	25–1
Four of a kind	15–1
Full house	9–1
Straight	6–1
Three of a kind	3–1
Two pairs	1–1

The probability of losing a Bonus bet is the probability of a 5-card hand of one pair or lower, which is

$$\frac{72,120}{\binom{28}{5}} \approx .7338.$$

The player/dealer, who banks the hand and pays off all winning players, holds an advantage of 5.39%.

The other prop bet is the Flop bet, which pays off based simply on the 3-card flop. There are $\binom{28}{3} = 3276$ possible flops. This bet pays 10–1 on a 3-card straight flush or 3 of a kind, 4–1 on a flush, 2–1 for a straight, and even money on a pair of jacks or higher. The probabilities of the various winning hands are these:

$$P(\text{Straight flush}) = \frac{20}{3276} \approx .0061.$$
$$P(\text{3 of a kind}) = \frac{28}{3276} \approx .0085.$$
$$P(\text{Flush}) = \frac{4 \cdot \binom{7}{3} - 20}{3276} \approx .0366.$$
$$P(\text{Straight}) = \frac{5 \cdot 4^3 - 20}{3276} \approx .0916.$$
$$P(\text{Jacks or better}) = \frac{4 \cdot \binom{4}{2} \cdot 24}{3276} \approx .1758.$$

The advantage to the player/dealer on the Flop bet is then 2.93%.

2 Way Winner

Deck composition:	53 cards
Hand size:	7 cards
Community cards:	5
Betting rounds:	1
Required bets:	Ante ×2
Optional bets:	Play ×2, 7-Card Bonus

2 Way Winner is a game from the Hollywood Park Casino in Inglewood that combines Texas hold'em and blackjack. Since California gaming laws do not permit blackjack-style games where the target number is 21, a joker which may be counted as 2 or 12 is added to the deck, and the target number is 22. The best hand is a Natural 22 consisting of the joker and a 10 or face card; the second-highest hand is a Joker 22, which is any hand of 22 containing the joker. 2 Way Winner uses a player/dealer, but all cards are dealt by a card room employee.

Players make equal Ante bets on poker and blackjack betting spaces. Play begins with 2 cards dealt to each player, face down, and two to the player/dealer, one face up and one face down. At this point, the game looks like blackjack. Player cards are dealt face down because knowledge of other players' hands might be beneficial in making the blackjack vs. poker decision. Based on the 3 cards they can see, each player must decide whether to play blackjack or poker against the player/dealer—a player cannot choose to play both games. Players may also fold at this point, forfeiting 1 of their 2 Ante bets; this is akin to surrendering in blackjack. Blackjack players must match their Ante bet with a blackjack Play bet, while poker players may bet either their Ante bet or double their Ante bet as a poker Play bet.

- If the player chooses blackjack, play proceeds under standard blackjack rules. The player/dealer checks for a natural 22 if her upcard is a joker or ten-count card; if her hand is a natural 22, all players lose. If not, players have the options to hit, stand, surrender, double down, and split a pair (but only once). Once all blackjack players' hands are resolved, the player/banker finishes her hand. Player/bankers must hit on soft 17 (a hand totaling 17 containing an ace counted as 11 or the joker counted as 12) or lower and must stand on hard 17 or higher. Winning blackjack players are paid 6–5 on their blackjack bets if they hold a natural 22 and 1–1 otherwise. A winning blackjack hand pushes the poker Ante bet.

- When the blackjack bets are resolved, play moves to poker against the remaining players. The first 5 cards of the dealer's blackjack hand, including the initial 2 cards, become the board for a round of Texas hold'em. If the player/dealer has more than 5 cards in the blackjack hand, any cards past the first 5 are removed from play. If the player/dealer's blackjack hand contains fewer than 5 cards, additional cards are dealt to fill out the board. The player/dealer then receives 2 face-down hole cards. In the poker phase of 2 Way Winner, the joker may be used only as a bug.

  There are no further rounds of betting; poker hands are compared and winners identified. Winners receive even money on their Poker Play and Poker Ante bets; the blackjack Ante bet pushes unless the player's winning hand is a full house or higher, in which case it also pays even money.

Example 7.40. Consider a 5-handed game, where the initial player hands are as follows:

- Robin: $K\spadesuit$ $6\diamondsuit$
- Chris: $Q\diamondsuit$ $7\clubsuit$
- Terry: $K\clubsuit$ $2\diamondsuit$
- Sandy: $K\heartsuit$ $Q\clubsuit$
- Pat: $A\heartsuit$ $3\diamondsuit$

The player/dealer's faceup card is the $T\spadesuit$. Since the dealer has a strong upcard and Robin and Chris have stiff hands, they opt to play poker. In an ordinary blackjack game with 21 as the target, Terry would ordinarily make the same choice if standard Las Vegas blackjack rules were in force, but since the target is 22, a starting hand of 12 is strong, so Terry chooses blackjack. Sandy and Pat also hold strong blackjack hands and choose blackjack. Terry hits and draws the $T\heartsuit$ for an unbeatable 22. Sandy stands on 20 and Pat hits soft 14, drawing the $7\heartsuit$ for 21. The player/dealer turns over the $J\diamondsuit$ for a 20, so Sandy pushes while Pat and Terry win.

The first two community cards are $T\spadesuit$ $J\diamondsuit$, and three more community cards are dealt: $2\heartsuit$ $J\clubsuit$ $3\clubsuit$. Robin and Chris both hold a pair of jacks. The player/dealer receives two hole cards: 8♠ and 5♠. Both poker players win based on their highest non-paired card: Robin's king and Chris' queen beat the player/dealer's T♠. All blackjack Ante bets push. ■

In standard single-deck blackjack, the probability of a natural 21 is

$$2 \cdot \frac{4}{52} \cdot \frac{16}{51} = \frac{32}{663} \approx \frac{1}{21}.$$

By contrast, the probability of a natural 22 in 2 Way Winner is much lower:

$$2 \cdot \frac{1}{53} \cdot \frac{16}{52} = \frac{8}{689} \approx \frac{1}{86}.$$

2 Way Winner offers an optional bet, 7-Card Bonus, which is based on the poker hand formed by the player's first two cards and the 5 community cards. This wager must be made before the cards are dealt and remains active even if the player making it chooses to play blackjack. In addition to the highest-ranked 5-card poker hands, special payoffs are available for 7-card straight flushes, either with or without the joker.

Example 7.41. Find the probability that a player's poker hand is a 7-card straight flush without the joker, which pays 5000–1 if the 7-Card Bonus bet has been made.

We are looking at all 7 cards, and it doesn't matter how they are distributed between the player's hand and the community cards. As we have done previously, we count 7-card straight flushes by counting the ways to pick the lowest card. Since aces count both high and low in straights and straight flushes, the 32 cards from ace through 8 fit this criterion.

The probability of a natural 7-card straight flush is then

$$\frac{32}{\binom{53}{7}} = \frac{32}{154,143,080} \approx \frac{1}{4,816,971}.$$

■

Table 7.54 shows the complete pay table for 7-Card Bonus.

7.7 Exercises

Answers to starred exercises begin on page 339.

7.1.⋆ In 2019, the game *Hold'em 88* debuted as a near-copy of Ultimate Texas Hold'em with an additional prop bet, the 88 Bonus bet. This bet was based

TABLE 7.54: 2 Way Winner: 7-Card Bonus bet pay table.

Winning hand	**Payoff odds**
Natural 7-card straight flush	5000–1
7-card straight flush with joker	750–1
5 aces	250–1
Royal flush	100–1
Straight flush	50–1
4 of a kind	20–1
Full house	5–1
Flush	4–1
3 of a kind	3–1
Straight	2–1

on the number of 8s appearing among the community cards and the player's hand, and paid off as in Table 7.55.

Calculating the various probabilities involved in the 88 Bonus bet requires that the community cards and a player's hole cards be considered separately. For example, the probability of losing this bet is simply the chance of drawing no 8s, or only one 8, among the 7 cards, which is

$$\frac{\binom{48}{5} \cdot \binom{43}{2} + \binom{48}{4} \cdot \binom{44}{2}}{\binom{52}{5} \cdot \binom{47}{2}} = \frac{2,282,501,232}{2,809,475,760} \approx .7797,$$

or approximately 78%. The first term in the numerator of this fraction is the number of ways to deal 7 cards, divided 5–2 between the community and hole cards, with no 8s. The second term accounts for the number of ways to deal one 8 among the community cards and none in the hole. We need not count the reverse situation where the player has a single 8 in the hole and no 8s appear among the community cards, since this hand is a winning hand that pays 2–1.

TABLE 7.55: Hold'em 88: 88 Bonus pay table.

Winning hand	**Payoff odds**
Four 8s with pocket 8s	200–1
Four 8s, any other way	100–1
Pocket 8s	30–1
Three 8s	20–1
Two 8s	4–1
One pocket 8	2–1

a. Find the probability that exactly two of the 5 community cards are 8s, which guarantees each player making this bet at least a 4–1 payoff.

b. Find the probability of drawing pocket 8s.

c. Compute the two probabilities of holding four 8s: one with pocket 8s and one without.

d. What is the HA of the 88 Bonus bet?

7.2.⋆ Another UTH variation, *Lucky 8s hold'em*. offers an optional *8s Fortune* prop bet that pays off if at least two 8s are dealt among the bettor's hole cards, community cards, and dealer's hole cards. The pay table is shown in Table 7.56.

TABLE 7.56: Lucky 8s Hold'em: 8s Fortune pay table.

Number of 8s	Payoff odds
4	200–1
3	20–1
2	3–1

If 2 or more 8s are dealt on the board, then every player making this bet wins.

a. Derive the PDF for the number of 8s in 9 cards.

b. What is the probability that an 8s Fortune bet loses?

c. Find the house advantage of the 8s Fortune bet.

7.3.⋆ Find the house advantage of a Player Hole Cards bet at All-in hold'em.

7.4.⋆ Find the HA of Big Raise Poker if small pairs also push instead of losing.

7.5.⋆ If Big Raise Poker pays even money on medium pairs and low pairs push, show that the resulting game is essentially a fair game, with a house advantage between −.1% and .1%.

7.6. Suppose that the dealer qualifying hand in CSP was raised to "any pair". Explain why the optimal player strategy should then be "Call on every hand".

7.7.⋆ Some Caribbean stud poker tables offer a $1 Progressive Payout prop bet. At the Borgata Casino in Atlantic City, this bet pays off when the player's hand is a flush or higher, regardless of whether or not it beats the dealer. Table 7.57 shows the payoffs.

TABLE 7.57: Caribbean Stud Poker: Progressive Payout pay table.

Hand	**Payoff odds**
Royal flush	Jackpot
Straight flush	10% of jackpot
4 of a kind	\$500
Full house	\$100
Flush	\$50

Find the probability of winning the Progressive Payout bet.

7.8.* Find the minimum jackpot J for which the CSP Progressive Payout prop bet described in Exercise 7.7 has a player edge.

7.9.* Bob Stupak's Vegas World (now the Strat), located on the north end of the Las Vegas Strip, offered a carnival game in the 1980s called *Vegas World Stud Poker*. This game was essentially Caribbean Stud Poker under a different name. The game included an optional \$1 Jackpot bet that was based on the player's hand and paid off according to Table 7.58.

TABLE 7.58: Vegas World 5-Card Stud: Jackpot bet pay table.

Hand	**Payoff**
Royal flush	\$10,000
Straight flush	\$1000
4 of a kind	\$500
Full house	\$100
Flush	\$50

Find the house advantage of this \$1 bet.

7.10.* The lowest set of 3 hole cards in Hickok's Six Card containing a pair is 223. Should you double your initial bet when holding these cards?

7.11.* On page 241, we saw that the 3-card Hickok's Six Card hand $T\clubsuit\ 6\diamondsuit\ 2\diamondsuit$ should be folded. Find the probability that the community cards fail to produce a hand that wins or pushes.

7.12.* Find the probability that a dealt Pai Gow Poker hand is a 7-card straight flush that includes the joker.

7.13.* An alternate pay table for the Pokara bonus bet is shown in Table 7.59.

How does the dealer advantage with this pay table compare to the edge given by Table 7.50 (page 271)?

7.14.* Find the probability that the dealer qualifies in a hand of Triple Action Poker.

TABLE 7.59: Pokara: Alternate Bonus bet pay table.

Winning hand	Payoff odds
A-A-A Hearts	500–1
Trip Flush	60–1
Royal Flush	30–1
Straight Flush	20–1
Three of a kind	10–1
Flush	3–1

High Country Poker

Deck composition:	51 cards (Dealer starts with $A♠$)
Hand size:	7 cards
Community cards:	5
Betting rounds:	2
Required bets:	Ante, Play
Optional bets:	Raise

High Country Poker was introduced in 1999, just ahead of the great 21st century Texas hold'em boom [19]. Players face off with the dealer to make the best 5-card hand from 2 hole cards and 5 community cards. The game is dealt from a 51-card deck, since the dealer always receives the $A♠$ as a hole card. Since this is known to players, only one dealer card is concealed before betting.

7.15.⋆ How has the probability that the dealer's 5-card hand is a pair of aces changed by this rule?

After making an Ante bet and a Play bet of double the ante, and viewing their hole cards, players are given several betting options. Any player holding one or two deuces may fold her hand for an immediate payoff. The Ante and Play bets are returned. A player folding a single deuce receives a 1–1 payoff on the Ante; if a pair of deuces is folded, the payoff is 5–1 on the Ante.

7.16.⋆ Optimal strategy for High Country Poker calls for folding any hand with a deuce and collecting the payoff [19]. Find the probability that a player is dealt a hand with 1 or 2 deuces.

Players who do not fold may play on for the 3 units at risk, or may make a Raise bet of 1–3 times their original Ante bet. After the 5 community cards are dealt, all at once rather than in 3 separate deals as in Texas hold'em, players and dealer make their best 5-card hand, and the showdown follows. The higher hand wins, and the player wins on all ties, with an even-money payoff on all bets in play.

7.17.* A player holding a pair other than deuces should make a 3× raise and play on to the showdown. Find the probability that a player is dealt a pair other than deuces.

High Country Poker pays bonuses for high-ranking player hands, called High Bonus hands. Table 7.60 shows the payoffs as multiples of the player's Ante bet.

TABLE 7.60: High Country Poker: High Bonus pay table [19].

Winning hand	**High Bonus payoff**
Royal flush	300–1
Straight flush	30–1
Four of a kind	10–1
Full house	5–1

7.18.* Find the probability that a High Country Poker player holds four of a kind.

7.19.* Find the probability that a High Country Poker player holds a royal flush.

Chapter 8

Video Poker

While Texas hold'em has come to dominate both live play in casinos and televised play in poker tournaments, *video poker* is most often a game of five-card draw poker played on an electronic terminal, much like a modern slot machine or video game (Figure 8.1). Just like a slot machine, players aim for the best hand as measured by a fixed pay table rather than seeking to beat another person's or electronic opponent's hand.

FIGURE 8.1: Video poker machines at Harrah's Casino, New Orleans, Louisiana [94].

Video poker involves player skill, making it one of the 4 common casino games (with blackjack, live poker, and sports betting) where a gambler can legally turn the house advantage into a player advantage by his or her actions. This rare player advantage is achieved by carefully choosing the right game and the right pay table for that game, using perfect strategy for the chosen game, and collecting players' club points based on one's gambling that can be redeemed for casino perks including free meals, free game play, or other valued considerations.

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The origins of video poker may be traced back to all-mechanical games dating back to the late 19th century. The first poker machine, developed by the Sittman and Pitt Company of Brooklyn, New York, was called Flip Card Poker: a mechanical version of 5-card stud poker similar to a slot machine. Players inserted a nickel into the machine and pulled a lever which sent 5 reels, each bearing 10 playing cards, spinning. When the reels stopped, the poker hand displayed determined the amount of any payoff [143]. Since gambling was largely illegal in America at the time, one official pay table, Table 8.1, listed payoffs in cigars, which were dispensed by bartenders to winning players.

The possibility that a player-friendly bartender might convert cigars to cash was not fully addressed by the machine's inventors.

TABLE 8.1: Sittman & Pitt Flip Card Poker machine pay table [57].

Hand	**Payoff**
Royal flush	100 cigars
Straight flush	25 cigars
4 of a kind	15 cigars
Full house	10 cigars
Flush	5 cigars
Straight	5 cigars
3 of a kind	3 cigars
2 pairs	2 cigars
Pair of kings or aces	1 cigar

The design of the game with 10 cards on each of 5 reels cut the number of possible hands from 2,598,960 to $10^5 = 100,000$. More importantly to players, 2 cards had to be removed from the deck to get down to the 50-card limit. Most commonly, the $T\spadesuit$ and $J\heartsuit$ were omitted, which meant that 2 of the possible 4 royal flushes—and the 100-cigar payoff that went with them—were impossible. Despite this, the probability of a royal flush increased, from 1 in 649,740 to 1 in 50,000.

Modern video poker uses computerized random number generators (RNGs) to deal acceptably random 5-card hands—"acceptably random" because the random numbers are generated by a fixed algorithm and are not truly random. The algorithm is sufficiently fast, generating thousands of numbers per second, that any attempt to "decode" the sequence and exploit the fact that it repeats every 32 billion or so cycles cannot succeed. The numbers are unpredictable enough that humans cannot deduce any pattern, which makes them good enough for computerized games of chance.

The earliest computerized video poker games would generate 10 cards. Five would be displayed to the player, and each of those had a hidden card behind it that would be revealed if the covering card was discarded. This method of generating poker hands was regarded by some observers as unfair to the player. If, for example, the player held $\diamondsuit 5678$ $\clubsuit K$ and discarded the king hoping to complete the straight flush, their chance of succeeding would be 0% if the $\diamondsuit$4 and $\diamondsuit$9 were both hidden behind the exposed diamonds. It was, of course, 100% if either the $\diamondsuit$4 or the $\diamondsuit$9 had been dealt behind the king. The probability of this latter event is $\frac{2}{47}$, the same as the probability of drawing a straight flush if the fifth card is not determined until the Draw button is pushed and the computer consults its RNG.

While the probability of completing the straight flush may well have been 2 in 47 regardless of the way that the replacement card was determined, the perception that the 10-card deal was unfair to players, together with improvements in computing technology, led to continuously-shuffled decks. While the player is making up his or her mind how to play the initial 5 cards, the machine continues generating random numbers. Once the player presses the "Draw" button, the computer retrieves the next 1–5 random numbers, determines the card or cards to which they correspond in the depleted deck with the initial 5 cards absent, and displays the final hand.

Video poker machines that operate this way are known as Class III gaming machines, a group that includes most slot machines as well as devices that play familiar casino games such as blackjack, poker, craps, and roulette in a manner akin to their live equivalents. Some gaming machines in Florida, Idaho, Washington state, and elsewhere appear to be video poker machines, but are in reality Class II games, where the outcome is determined by a random number generator regardless of the player's choices. These games can be identified by the presence of an electronic bingo card on the display. From a programmer's perspective, a Class II game is a game of bingo, with a fixed outcome determined before the cards are dealt and not subject to change based on a player's choices. From a regulator's perspective, since bingo may legally be offered by all Native American tribes located in states where any form of gambling is legal (all states except Hawaii and Utah), these games are legal in tribal casinos where standard video poker or slot machines may not be. To a gambler, the differences between Class II and Class III gaming machines may be little more than cosmetic.

A gambler playing a Class II poker machine can be dealt a straight flush, capriciously discard all 5 cards, and always be dealt another straight flush, because the playoff is fixed and is independent of player choices [119]. If you're dealt 4 of a kind on a Class II game and discard 3 of the 4 matching cards, the draw will restore a different 4 of a kind. If you discard 2 cards to 4 of a kind, leaving no way for a Class III machine to restore any 4 of a kind, some Class II machines activate an animated genie who "magically" restores the winning hand. Playing a hand of video poker on one of these machines is equivalent to buying an instant lottery ticket with a predetermined result.

(Class I gaming, as codified in the USA in the Indian Gaming Regulatory Act which legalized casinos run by Native American tribes on reservation lands in 1988, encompasses traditional tribal games and is explicitly not regulated by the federal government. Class I games are not usually found in casinos.)

With modern computers, millions of numbers may be generated and discarded while players make their decisions, and millions more are easily available. With the time between one number and the next less than .001 seconds, there is no way for a player to affect the next random number—for better or for worse—by any attempts to time their push of the button. Hesitating before committing to the Draw button merely induces a random change in the number generated and the subsequent card or cards that are dealt.

8.1 Basic Gameplay

In the most elementary form of modern video poker, players insert cash into a machine and press a "Deal" button. A five-card hand is dealt, and the player chooses which cards to hold and which to discard and replace with fresh cards from the remainder of the deck. Players may hold all 5 initial cards, discard them all, or choose any subset of their hand to replace. Since there are 2 choices for each card—keep it or toss it—there are $2^5 = 32$ ways to play a dealt video poker hand.

The first challenge to a video poker player's skill comes in selecting a machine or a game within a machine, for different machines with varying pay tables can carry different house edges even as they look identical. Some video poker machines offer multiple games in the same box, and careful selection of the best game is key to success at video poker.

Jacks or Better

The simplest video poker machines pay off on any final hand of at least a pair of jacks, and so are called Jacks or Better machines. In the 1970s, the earliest electronic video poker games set two pairs as the lowest winning hand, but they attracted little player interest until a hand of a pair of jacks through aces was declared a push rather than a player loss, and the initial wager returned.

A sample pay table is shown in Table 8.2. Payoffs depend on the number of coins (betting units) wagered; the denomination of that coin could be anything from a penny to $25 or more. All payoffs here are "for 1", meaning that the player's original wager is returned as part of the payoff. Accordingly, a pair of jacks through aces simply breaks even.

In reading this table, one notes that when betting five coins, the payoff on every hand but one is simply five times the payoff with a single coin at risk. Leaving out this exception, choosing the number of coins to play would simply be a matter of assessing one's concern for variance, as the expected return as a percentage of the wager is not affected. The exception is the royal flush, and it's a big one. Since approximately 2% of the payback percentage on most video

TABLE 8.2: Jacks or Better video poker pay table.

	—— Coins wagered ——				
Poker hand	**1**	**2**	**3**	**4**	**5**
Royal flush	250	500	750	1000	**4000**
Straight flush	50	100	150	200	250
4 of a kind	25	50	75	100	125
Full house	9	18	27	36	45
Flush	6	12	18	24	30
Straight	4	8	12	16	20
3 of a kind	3	6	9	12	15
2 pairs	2	4	6	8	10
Pair, jacks-aces	1	2	3	4	5

poker machines comes from royal flushes betting the maximum, or playing "max coins" for short, optimal video poker strategy relies on the determined pursuit of royal flushes. A player who doesn't draw to an incomplete royal flush whenever the strategy tables advocate doing so is giving up on a meaningful part of the payoff: reducing a 98% return machine to a 96% one, for example. A video poker player seeking the best possible return must bet max coins. This is one way in which the pay table on a video poker machine dictates some strategy decisions before the cards are even dealt.

Video poker pay tables are commonly described in brief by the payoff for a one-coin bet on a full house and a flush; so this table would be called a *9/6* game. A 9/5 machine, with all other payoffs the same, is better for players than a 9/5 machine, which is in turn better than an 8/5 machine. The house advantage of a video poker machine can be calculated using the frequencies of each possible hand, assuming perfect player strategy (Section 8.3), combined with the game's payoff schedule. This advantage is commonly expressed as a game's *payback percentage*: the percentage of the money wagered that is returned to players in the long run. If this percentage should exceed 100%, then a gambler employing perfect strategy has an edge over the casino. This percentage indicates how much of the money played in the machine is paid out to gamblers. For the payoffs in Table 8.2, this return percentage against a player using optimal strategy is 99.54%, and so the house advantage, frequently called the "hold percentage" in video poker, is .46%. If the payoff for a royal flush betting 5 coins is raised to 4700 coins, this game has a 99.8% return and a .2% hold percentage [84].

How is a given machine's payback percentage computed? When the pay table is combined with the recommended strategy for a game, it is possible to find the probability of each final hand—provided that the player faithfully uses the optimal strategy. These probabilities can be multiplied by the corresponding payoffs to find the expected value of a wager; see Table 8.3 for the calculations for a 9/6 Jacks or Better machine. Summing the rightmost

TABLE 8.3: 9/6 Jacks or Better expectation: 99.54% return percentage [131].

Poker hand	**Probability**	**Payoff**	**Expectation**
Royal flush	2.476×10^{-5}	4000	.0990
Straight flush	1.093×10^{-4}	250	.0273
4 of a kind	.0024	125	.2953
Full house	.0115	45	.5178
Flush	.0110	30	.3304
Straight	.0112	20	.2245
3 of a kind	.0746	15	1.1194
2 pairs	.1299	10	1.2987
Pair, JOB	.2128	5	1.0638
Nothing	.5556	0	.0000
		Total:	4.9763

column in Table 8.3 gives a player return of 4.9763 coins per 5-coin wager, leading to the low house edge of .46% quoted above.

The payback percentage of 99.54% for a 9/6 game drops to 97.23% for an 8/6 game, as shown in Table 8.4. The probabilities of some hands differ

TABLE 8.4: 8/5 Jacks or Better expectation: 97.23% return percentage [131].

Poker hand	**Probability**	**Payoff**	**Expectation**
Royal flush	2.489×10^{-5}	4000	.0996
Straight flush	1.077×10^{-4}	250	.0269
4 of a kind	.0024	125	.2954
Full house	.0115	40	.4603
Flush	.0109	25	.2726
Straight	.0112	20	.2247
3 of a kind	.0746	15	1.1194
2 pairs	.1299	10	1.2987
Pair, JOB	.2128	5	1.0638
Nothing	.5556	0	.0000
		Total:	4.8614

slightly between Tables 8.3 and 8.4 because the pay tables have changed, and with them the optimal strategies.

In order to enjoy this low house advantage, it is essential that the player follow the game's recommended strategy *exactly*; any deviation from optimal play will invariably increase the hold percentage. In light of the increased payoff for a royal flush when betting max coins, a video poker player's drawing strategy should be tilted toward—though not myopically obsessed with—pursuing royal flushes, which have a probability of approximately 1 in 40,000 on a video poker machine. This value depends on the optimal strategy, which is influenced by the pay table, and so varies slightly among games. Contrast this with the probability of 1 in 649,740 of receiving a dealt royal flush, and the value of being allowed to discard cards and replace them in the draw is clear.

How do we arrive at $1/40,000$ as the approximate probability of a royal flush in a draw game? Consider 3 ways to get a royal flush at video poker:

- The probability of a dealt royal flush is the same as it is in any 5-card poker game:
$$P(\text{Dealt royal flush}) = \frac{4}{\binom{52}{5}}.$$
- Another way to get a royal flush is to draw 1 card to an initial 4-card royal and get the fifth card. This event has a probability of
$$\frac{20 \cdot 47}{\binom{52}{5}} \cdot \frac{1}{47} = \frac{20}{\binom{52}{5}}.$$
- The probability of a royal flush when drawing 2 cards to a dealt 3-card royal is
$$\frac{4 \cdot \binom{5}{3} \cdot \binom{47}{2}}{\binom{52}{5}} \cdot \frac{1}{\binom{47}{2}} = \frac{40}{\binom{52}{5}}.$$

We might reasonably continue in this vein, considering 3-card draws to 2-card royals, but this ignores the reality that optimal video poker strategy for many games holds that a hand such as $A\spadesuit\ Q\spadesuit\ 9\clubsuit\ 9\heartsuit\ 5\diamondsuit$ is best played by drawing 3 cards to the pair of 9s rather than pursuing a royal flush, so many 2-card royals—and indeed, some 3-card royals such as $K\diamondsuit\ Q\diamondsuit\ T\diamondsuit\ T\clubsuit\ T\heartsuit$— will be discarded as the player tries to improve the starting hand differently, here by drawing 2 cards to the three 10s.

Adding these 3 probabilities gives
$$P(\text{Royal flush}) \approx \frac{64}{\binom{52}{5}} = \frac{1}{40,608.75},$$
which is within 1.5% of $1/40,000$.

We have also omitted some lower-probability events that might be played out, such as receiving a royal flush on the draw after discarding all 5 initial cards. A video poker hand where all 5 cards should be discarded is sometimes called a *razgu*; its probability is about 3.5% [47]. Assuming that the razgu contains no 10s, the probability of drawing a royal flush after discarding a razgu is

$$\frac{4}{\binom{47}{5}},$$

making the probability of receiving a 10-free razgu and then drawing a royal flush approximately

$$(.035) \cdot \frac{4}{\binom{47}{5}} \approx \frac{1}{10,959,707}.$$

If an initial Jacks or Better hand contains a single jack, queen, king, or ace, with no straight or flush possibility, optimal strategy calls for the player to hold that card and draw 4 new cards, so no razgu can contain a card higher than a 10.

Omitting this and similar terms does not change the fact that the common estimate $P(\text{Royal flush}) \approx {}^{1}\!/_{40,000} = 2.5 \times 10^{-5}$ is very accurate. Using this as the probability of a royal flush allows some simple calculations about the frequency of royals in actual gameplay—for video poker machines do not know the laws of probability and so do not understand that they're supposed to give the player a royal flush every 40,000 hands.

Poisson Distribution

An event like a royal flush, with probability $p = .000025$, certainly qualifies as a rare event. While we could compute the probabilities of k royal flushes across a number n of video poker hands using the binomial distribution—after all, every poker hand is either a royal flush or it's not—a tiny probability like p, especially when raised to a large power, may cause underflow errors in calculations where very small numbers may evaluate to 0, incorrectly. An alternate way to model rare events is given by the *Poisson* distribution.

Definition 8.1. A random variable X has a *Poisson distribution* if the following criteria are met:

1. X counts the number of occurrences of an event in a fixed interval of time or other quantity. Note that this means that X is a discrete random variable that can only take on nonnegative integer values.

   In more general settings, this focus on a fixed time interval rather than a fixed number of trials distinguishes Poisson RVs from binomial RVs. For our purposes, the "fixed time interval" can simply count a number of video poker hands.

2. Occurrences of the event being counted are independent.

3. It is impossible for two occurrences of the event to happen simultaneously.

   This condition addresses the challenge inherent in counting discrete events occurring across a continuous time interval. For example, when counting the number of radioactive atoms that decay in a 1-minute interval, a Poisson model assumes that two atoms cannot decay at the exact same instant.

4. The average rate at which events occur is independent of whether or not the event has occurred.

Example 8.1. Let X count the number of royal flushes in a given number of video poker hands. The successive hands are independent, and it is not possible to receive 2 royal flushes in a single hand. Since we assume a fixed probability $P(\text{Royal flush}) = \frac{1}{40,000}$, which does not change after a single hand regardless of whether or not the result of the hand is a royal flush, X has a Poisson distribution. ■

If X is the number of successes in n trials, a Poisson random variable X has PDF

$$P(X = k) = \frac{e^{-\mu}\mu^k}{k!},$$

where μ (the Greek letter mu) is the mean number of expected successes and $e = 2.71828...$ is the irrational base of the natural logarithm. For convenience when the exponent z is a complicated algebraic expression, we may write $\exp(z)$ for e^z.

In the cases we shall consider here, this formula may be regarded as the limit of a binomial distribution as $n \to \infty$. Under the assumption that the probability of success is constant from trial to trial, $\mu = np$: the mean of a binomial RV with parameters n and p.

If we deal and play out 40,000 hands using optimal strategy for Jacks or Better, then $\mu = 1$, but the probability of exactly 1 royal flush among those hands is

$$P(X = 1) = \frac{e^{-1}1^1}{1!} = e^{-1} \approx .3679,$$

less than 40%. The most likely outcomes are 0 royals or 1 royal, since

$$P(X = 0) = e^{-1} = P(X = 1)$$

and $P(X = 2)$ is half this value, with $P(X = k) = \frac{e^{-1}}{k!}$ decreasing sharply as k increases past 2. The probability of 9 royals in 40,000 hands is approximately 1 in 1 million, while the probability of 10 or more royal flushes is close to 1 in 9 million.

The probability of 1 or more royal flushes in n hands is

$$P(X \geqslant 1) = 1 - P(X = 0),$$

where the Poisson PDF is computed with $\mu = np = \dfrac{n}{40,000}$. This translates to

$$P(X \geqslant 1) = 1 - \exp\left(-\frac{n}{40,000}\right).$$

In 80,000 hands, where we would expect an average of 2 royals, we have

$$P(X \geqslant 1) = 1 - \frac{e^{-2} \cdot 2^0}{0!} = 1 - e^{-2} \approx .8647,$$

leaving a probability of about .1353 of no royal flushes.

Example 8.2. How many hands must be played to have a 95% chance of at least one royal flush?

We need to solve the equation

$$1 - \exp\left(-\frac{n}{40,000}\right) = .95$$

for n. We have

$$\begin{aligned}
\exp\left(-\frac{n}{40,000}\right) &= .05. \\
-\frac{n}{40,000} &= \ln .05. \\
n &= -40,000 \cdot \ln .05.
\end{aligned}$$

Rounding up to the nearest integer gives $n = 119,830$. After this many hands, the expected number of royals is nearly 3. ■

8.2 Variations

Video poker rose in popularity when computer graphics advanced enough to allow images of playing cards to be displayed on a video monitor, beginning in the late 1970s [143]. In the ensuing decades, a wide assortment of new games was introduced in an attempt to cash in on this new game idea. While the market settled down somewhat and many games have fallen out of favor in casinos, some of these innovations are interesting both mathematically and for their strategy implications.

Bonus Poker

Bonus Poker is a video poker variant that pays extra on some four-of-a-kind hands. These bonuses are paid for by lower payoffs for some lesser hands, the goal being to generate more excitement in the form of higher payoffs. Game balance is preserved with smaller payoffs at the lower end of the pay table, with the net result being a larger variance. A sample Bonus Poker pay table returning 99.16% is shown in Table 8.5.

TABLE 8.5: Bonus Poker pay table [131].

	—— Coins wagered ——				
Poker hand	**1**	**2**	**3**	**4**	**5**
Royal flush	250	500	750	1000	4000
Straight flush	50	100	150	200	250
4 aces	80	160	240	320	400
4 of a kind: 2s–4s	40	80	120	160	200
4 of a kind: 5s–kings	25	50	75	100	125
Full house	8	16	24	32	40
Flush	5	10	15	20	25
Straight	4	8	12	16	20
3 of a kind	3	6	9	12	15
2 pairs	2	4	6	8	10
Pair, jacks or better	1	2	3	4	5

In the lingo of Jacks or Better video poker, this would be an 8/5 game; the bonus payoffs on some four of a kinds restore the return percentage to a value above 99%.

Suppose that a video poker player played 2,598,960 hands on a 9/6 Jacks or Better game at max coins and hit every possible 5-card hand exactly once. This player would hit all 624 four-of-a-kind hands and would receive $125 \times 52 = 78,000$ coins from those hands. The player who hit the same collection of four of a kinds in Bonus Poker would collect 102,000 coins, a 30.8% increase.

Our 9/6 Jacks or Better player would take in 168,480 coins from full houses, while her Bonus Poker counterpart would only receive 149,760 coins: 11.11% less. Similarly, the total payoff from flushes would decrease by 16.67% when playing Bonus Poker.

Quick Quads

Like Bonus Poker, *Quick Quads* changes how selected hands are valued; as a result, more hands are scored as four of a kind. In a hand with 3 deuces through 10s, two other cards whose ranks add up to the rank of the triple can be counted as a fourth card of that rank, and so a hand such as 99963 is paid off as four 9s. Aces count as 1 in Quick Quads. This rule change usually requires an additional investment of one coin per hand, so playing Quick Quads at max coins requires a 6-coin bet per line.

Quick Quads hands with 3 cards of rank X that count as four of a kind have the form $XXXA(X-A)$. If X is odd, there are

$$\binom{4}{3} \cdot \frac{X-1}{2} \cdot 4^2 = 32(X-1)$$

such hands. The factor $X-1$ counts the number of ways to choose A; we divide by 2 since choosing A is the same as choosing $X-A$. Since X is odd, $X-2$ is even and so dividing by 2 leaves a natural number.

Hands with even values of X require that hands that look like full houses but are scored as Quick Quads be considered separately; these have the form $XXX\frac{X}{2}\frac{X}{2}$. There are $\binom{4}{2} = 6$ ways to choose the pair and $\frac{X-2}{2} \cdot 4^2$ ways to select two nonmatching cards that add up to X. Taken together, we have

$$\binom{4}{3}\left[\left(\frac{X-2}{2}\right) \cdot 4^2 + \binom{4}{2}\right] = 32(X-1) - 8$$

hands with this composition.

Example 8.3. A Quick Quads hand of 3 deuces must be $222AA$. There are $\binom{4}{3} = 4$ ways to choose the 2s and $\binom{4}{2} = 6$ ways to pick the aces. Multiplying gives 24 hands, which is what we get from the formula above with $X = 2$. ■

Since the game charges a 1-coin per hand premium to the player to activate Quick Quads, the increased number of four-of-a-kind hands can be accommodated without reducing payoffs further down the pay table. In Table 8.6, we see a Quick Quads pay table on a 9/6 Jacks or Better game returning 99.61%.

TABLE 8.6: Quick Quads pay table [26].

	Coins wagered	
Poker hand	**1**	**6**
Royal flush	250	4000
Straight flush	50	250
4 of a kind	25	236
Full house	9	45
Flush	6	30
Straight	4	20
3 of a kind	3	15
2 pairs	2	10
Pair, jacks or better	1	5

Note the increased payoff on 4 of a kind, which would be 125 coins with a 5-coin wager.

Video Poker With Wild Cards

Joker Poker

Adding a joker to a 52-card video poker deck allows for stronger hands, just as it does in live poker. The pay table for video poker with a wild joker, frequently called Joker Poker, changes to accommodate this extra card:

- 5 of a kind is a possible hand.
- Royal flushes are separated into "natural" royals, without the joker, and "joker" royals, which contain it. Natural royals, which still number 4, are the best-paying hand, and usually the only hand carrying a bonus payoff at the 5-coin level.

  Other hands are not so separated: all full houses, for example, are considered equal whether they contain the joker or not.
- The lowest hand that pays off changes. In some games, the pay table starts at 2 pairs, as in Table 8.7. In others, the machine pays off on a pair of kings or aces [84].

TABLE 8.7: Joker Wild video poker pay table.

	—— Coins wagered ——				
Poker hand	**1**	**2**	**3**	**4**	**5**
Natural royal flush	500	1000	1500	2000	4000
5 of a kind	100	200	300	400	500
Joker royal flush	50	100	150	200	250
Straight flush	50	100	150	200	250
4 of a kind	20	40	60	80	100
Full house	10	20	30	40	50
Flush	6	12	18	24	30
Straight	5	10	15	20	25
3 of a kind	2	4	6	8	10
2 pairs	1	2	3	4	5

Example 8.4. How many possible royal flushes with the joker are possible?

As we know, there are 4 royal flushes. In each of them, any one of the 5 cards can be replaced by the joker, giving a total of 20 jokered royal flushes. ■

The probability of being dealt a royal flush with a joker is then

$$\frac{20}{2,869,685} = \frac{4}{573,937} \approx 6.9694 \times 10^{-6}.$$

The chance of a royal flush after drawing depends on the discard-and-draw strategy that a player uses. A player using the optimal strategy for Joker Poker can expect to hit a royal flush approximately once every 16,000 hands [84].

While this is 2.5 times more frequently than on a Jacks or Better machine, royals are still sufficiently rare that the Poisson distribution is a good choice for computing long-range probabilities. The number of hands n required to have a 95% chance of at least one royal flush is then given by

$$1 - \exp\left(-\frac{n}{16,000}\right) = 0.95,$$

which has solution $n = 47,932$ to the nearest integer.

Deuces Wild

By designating all 4 deuces as wild cards, video poker increases the excitement and the potential for higher hands beyond that reached with a single wild joker. With so many more good hands available, 3 of a kind becomes the lowest-paying hand. In Deuces Wild, 3 of a kinds can be sorted by their number of wild cards. Three-of-a-kind hands without wild cards were counted in Section 1.4; this count of 54,912 must be reduced by the number of hands holding 1 or 3 deuces. (A hand containing 3 of a kind and 2 deuces was counted in Section 1.4 as a full house and will play here as 5 of a kind.) There are

$$12 \cdot \binom{4}{3} \cdot \binom{11}{2} \cdot 4^2 = 42,240$$

3 of a kinds that contain no deuces.

A 3 of a kind with a single deuce will count the deuce as a card in the triple and have the form $2xxyz$; these number

$$12 \cdot \binom{4}{2} \cdot 4 \cdot \binom{11}{2} \cdot 4^2 = 253,440.$$

Finally, 3 of a kinds with 2 deuces look like $22xyz$, where $x > y > z$ and these three cards do not permit forming a flush or straight with the wild cards.

$$\binom{4}{2} \cdot \left[\binom{12}{3} - 55\right] \cdot (4^3 - 4) = 59,400.$$

The complete frequencies for dealt hands in Deuces Wild are shown in Table 8.8. While this table ranks the hands in their proper order by frequency, the Deuces Wild pay table retains the standard order of 5-card poker hands, adding 5 of a kind near, though not at, the top. Royals and five of a kinds are divided into two subcategories: natural royal flushes without any deuces pay better than royals with one or more deuces, and 5 of a kind with 4 deuces outranks other five of a kinds.

TABLE 8.8: Deuces Wild initial hand frequencies.

Hand	**Count**
Royal flush	484
5 of a kind	672
Straight flush	2068
Full house	12,672
Flush	14,472
4 of a kind	31,552
Straight	62,232
2 pairs	95,040
3 of a kind	355,080
High card	799,680
Pair	1,225,008

The most lucrative Deuces Wild game is known to players as *Full-Pay Deuces Wild* (FPDW). The game gets its nickname from the fact that, with perfect play, it returns 100.76% to players. Its pay table is shown in Table 8.9.

TABLE 8.9: Full-Pay Deuces Wild video poker pay table.

	—— Coins wagered ——				
Poker hand	**1**	**2**	**3**	**4**	**5**
Natural royal flush	250	500	750	1000	4000
4 deuces	200	400	600	800	1000
Wild royal flush	25	50	75	100	125
5 of a kind	15	30	45	60	75
Straight flush	9	18	27	36	45
4 of a kind	5	10	15	20	25
Full house	3	6	9	12	15
Flush	2	4	6	8	10
Straight	2	4	6	8	10
3 of a kind	1	2	3	4	5

Table 8.10 shows an alternate Deuces Wild pay table. This version of Deuces Wild is sometimes called *Illinois Deuces*, and returns only 98.91%. The name comes from a state law that forbids electronic gaming machines that return more than 100% [63].

2,119,728 hands, approximately 81.6%, are nonpaying hands ranking 2 pairs or lower. Since 4 of a kind is considerably more likely with 4 wild cards in play, the payoff drops accordingly: from 20 for 1 in Jacks or Better to 4 for 1 in Illinois Deuces.

TABLE 8.10: Illinois Deuces pay table.

	— Coins wagered —				
Poker hand	**1**	**2**	**3**	**4**	**5**
Natural royal flush	800	1600	2400	3200	4000
4 deuces	200	400	600	800	1000
Wild royal flush	25	50	75	100	125
5 of a kind	15	30	45	60	75
Straight flush	9	18	27	36	45
4 of a kind	4	8	12	16	20
Full house	4	8	12	16	20
Flush	3	6	9	12	15
Straight	2	4	6	8	10
3 of a kind	1	2	3	4	5

As when moving from Jacks or Better to Joker Poker, the probability of a royal flush rises in Deuces Wild poker, to 1 in 600 when perfect strategy is followed [84].

Video Poker With Different Decks

Five Aces

Five Aces video poker, as the name suggests, is played with a 53-card deck including a fifth ace, the $A\bigstar$. As this is the only card in the $\bigstar$ suit, it is impossible to form any kind of flush in a hand with the $A\bigstar$. There is one way to get 5 aces, which makes it the highest-ranking hand.

Every hand from standard 5-card draw poker is a valid hand in Five Aces. Counting the hands in Five Aces is facilitated by separating them into the $\binom{52}{5} = 2,598,960$ hands that do not contain the $A\bigstar$ and the $\binom{52}{4} = 270,725$ hands that do contain it.

Example 8.5. There are 54,912 3-of-a-kind hands in a standard deck. The number of 3 of a kinds with the $A\bigstar$ is the sum of two quantities.

- Three-ace hands of the form $A\bigstar AAxy$, with $x \neq y$ and neither equal to A. There are
$$\binom{4}{2} \cdot \binom{12}{2} \cdot 4^2 = 6336$$
hands of this type.
- Hands containing the $A\bigstar$ and 3 cards of a different rank, which look

like $xxxyA\bigstar$, again with $x \neq y \neq A$. These number

$$12 \cdot \binom{4}{3} \cdot 44 = 2112.$$

Adding gives 8448 three of a kinds including the $A\bigstar$, for a total of 63,360 three of a kinds overall. ■

Five Aces is a variant on Jacks or Better. The order of the 10 types of paying poker hands is unchanged, as the addition of a fifth ace does not change the relative rank of any hands except that the unique 5-ace hand sits atop the pay table, with a probability of

$$\frac{1}{\binom{53}{5}} = \frac{1}{2,869,685},$$

and also carries a bonus when 5 coins are played.

Table 8.11 shows one pay table for Five Aces. Four of a kinds are separated

TABLE 8.11: Five Aces pay table.

	Coins wagered	
Poker hand	**1**	**5**
5 aces	500	6000
Royal flush	250	4000
Straight flush	100	500
4 aces	80	400
4 of a kind: 2s–4s	40	200
4 of a kind: 5s–kings	25	125
Full house	9	45
Flush	6	30
Straight	4	20
3 of a kind	3	15
2 pairs	1	5
Pair, jacks or better	1	5

out by ranks, with 4 aces being the most lucrative despite being more likely than any other four of a kind, and 2-pair hands pay only 1 for 1 instead of 2 for 1.

The $A\bigstar$ is part of

$$\binom{4}{2} \cdot 12 \cdot \binom{4}{2} + 12 \cdot \binom{4}{3} \cdot 4 = 624$$

additional full houses, but adding these to the 3744 full houses in a 52-card deck only gives 4368 full houses, which is fewer than the 5108 flushes in the deck. Full houses still rank higher than flushes even though the $A\bigstar$ is part of no flushes.

There are, of course, considerably more 1-pair hands that contain a pair of aces than any other pair. Recall that there are 84,480 one-pair hands of a given rank in a 52-card deck. Adding a fifth ace gives 56,320 more hands of a pair of aces and 5280 more hands with a pair of any specified lower rank, for a total of 140,800 pairs of aces and 89,760 pairs of any other rank.

Five Deck Poker

Five Deck Poker is a video poker innovation that deals each of the 5 cards in a player's hand from a separate 52-card deck. This greatly increases the number of possible hands, to $52^5 = 380,204,032$. Included among this number are new paying hands such as 5 of a kind and suited 5 of a kinds consisting of 5 copies of the same card.

There are 52 suited 5-of-a-kind hands, since once the card is selected, we must have 5 copies of it. Formal application of the Fundamental Counting Principle gives the number of suited 5 of a kinds as $52 \cdot 1 \cdot 1 \cdot 1 \cdot 1$: 52 ways to pick the leftmost card, and then 1 way to pick each of the others. Unsuited 5 of a kinds are counted by starting with the rank and removing the suited hands at the end. With 4 choices for the suit of each card, we have

$$13 \cdot (4^5 - 4) = 13,260$$

unsuited hands. The probability of a dealt unsuited 5-of-a-kind hand is

$$\frac{13,260}{380,204,032} \approx \frac{1}{28,273}.$$

With 5 full decks in play, royal flushes are counted by considering the number of permutations of the 5 cards. There are

$$4 \cdot {}_5P_5 = 4 \cdot 120 = 480$$

royal flushes. The probability of a dealt royal flush has fallen from the 1 in 649,740 in Jacks or Better to about 1 in 792,092.

A Five Deck Poker pay table is shown in Table 8.12. Note that some hands pay off more highly if they are suited hands: where all of the scoring cards are the same suit.

In a suited 4 of a kind hand, the 5th card need not be of the same suit as the 4 matching cards, and similarly for other suited hands with fewer than 5 scoring cards. A suited full house has the 3 of a kind of one suit and the pair of the same suit, possibly a different suit from the 3 of a kind. Suited 2-pair hands can have a different suit for each pair, as with $K\diamondsuit\ K\diamondsuit\ T\clubsuit\ T\clubsuit\ 8\diamondsuit$.

TABLE 8.12: Five Deck Poker pay table [84].

	Coins wagered	
Poker hand	**1**	**5**
Suited 5 of a kind	1000	10,000
Royal flush	250	4000
Straight flush	50	250
5 of a kind	50	250
Suited 4 of a kind	25	100
Suited full house	12	60
4 of a kind	10	50
Full house	6	30
Suited 3 of a kind	4	20
Flush	4	20
Straight	3	15
3 of a kind	2	10
Suited 2 pairs	2	10
2 pairs	1	5

The probability of receiving a pair in the first 5 cards is

$$\frac{\left[\binom{5}{2} \cdot 52 \cdot 4\right] \cdot \ {}_{12}P_3 \cdot 4^3}{52^5} = \frac{175,718,400}{380,204,032} \approx .4622,$$

but this includes many hands containing a suited pair among 5 cards of the same suit, which are rightly scored as flushes. These hands can be counted by simply replacing the 4s in the numerator above by 1s, which gives

$$\left[\binom{5}{2} \cdot 52 \cdot 1\right] \cdot \ {}_{12}P_3 \cdot 1^3 = 686,400,$$

exactly $1/256$ of the total. The probability of a one-pair non-flush hand is then

$$\frac{175,718,400 - 686,400}{380,204,032} \approx .4604.$$

Since each card is dealt from a separate deck, successive cards on the initial deal are independent. Using 5 independent decks changes the probabilities of improving hands on the draw, some for the better, some for worse.

- The probability of improving a dealt two-pair hand to a full house in a standard video poker game is $\frac{4}{47} \approx .0851$. In Five Deck Poker, this probability is $\frac{8}{51} \approx .1569$, close to twice as high.

- If the player draws 1 card to a 4-card flush, the probability of completing the flush is $\frac{13}{51} \approx .2549$, considerably better than the chance of $\frac{9}{47} \approx .1915$ of drawing the fifth card of the targeted suit in a standard video poker game.
- By contrast, the chance of competing a 4-card straight that's open at both ends, such as 6789, is only $\frac{8}{51} \approx .1569$, which is slightly less than the chance of $\frac{8}{47} \approx .1702$ one gets when drawing 1 card to a straight in Jacks or Better.

Video Stud Poker

While most video poker games are variations on draw poker, games in this section have attempted to adapt 5-card and 7-card stud poker to the video format.

Double Down Stud

Double Down Stud is a modification of 5-card stud poker into a video game. This change of game leads to some differences between Double Down Stud and video draw poker: chief among them are larger payoffs at Double Down Stud and the corresponding lower probability of winning hands due to the inability to discard and draw cards [125].

A hand of Double Down Stud begins when the player places a wager and receives one card. The machine then deals four community cards, three face up and one face down. Players are given the option to double their bet before the fourth community card is revealed. Though players must stick with their hands through the final community cards and do not have the option of folding, the pay table means that there is no 4-card hand without a chance of improving to a paying hand with the 5th card. This is implemented by paying off on a pair of 6s or higher: a 4-card hand with neither a pair nor any card above a 6 must be 2345, which can still draw out to a straight, or to a flush if the cards are suited. Winning poker hands are paid off according to Table 8.13.

The payoffs in Table 8.13 are "to 1", not "for 1", even though the mechanics of a Double Down Stud machine are the same as a draw poker machine. On a pair of 6s through 10s, a player's wager is returned.

Example 8.6. Suppose that the first 4 cards in a Double Down Stud hand are

$$Q\clubsuit\ 9\spadesuit\ Q\heartsuit\ 7\heartsuit.$$

What is the expected value of this hand?

The following outcomes are possible:

TABLE 8.13: Payoff table for Double Down Stud [125].

Poker hand	**Payoff odds**
Royal flush	2000–1
Straight flush	200–1
4 of a kind	50–1
Full house	11–1
Flush	8–1
Straight	5–1
3 of a kind	3–1
Two pairs	2–1
Pair, jacks or better	1–1
Pair, 6s through 10s	Push

Poker hand	**Ways to reach**	**Net payoff**
Three queens	2	3
Two pairs	6	2
Pair of queens	40	1

The expected return on this hand is

$$E = (3) \cdot \frac{3}{48} + (2) \cdot \frac{6}{48} + (1) \cdot \frac{40}{48} = \frac{61}{48} \approx \$1.27.$$

The choice is easy. You should double your bet and prepare for a win.

■

3-Way Action Poker

3-Way Action Poker allows a gambler the chance to play 3 games in quick succession.

- The first 5 cards are played as a 5-card stud poker hand, with payoffs starting at a pair of 3s.
- After the stud hand is resolved, players can then treat the hand as the initial hand in 5-card draw, discarding and drawing in an effort to improve the hand. The pay table is based on a game variant called *Double Double Bonus Poker*, which pays off on a pair of jacks or higher and offers different payoffs for different hands at the four-of-a-kind level. The top four-of-a-kind payoff is awarded for four aces with a 2, 3, or 4 as kicker.
- When the final draw hand is determined, 2 additional cards are dealt and the resulting 7 cards are played as a 7-card stud hand. Three of a kind is the lowest paying hand. Certain high-ranking 7-card hands are

recognized with larger payoffs; the top prize of 4000 for 1 is reserved for a hand consisting of 4 aces and three 2s, 3s, or 4s.

The game operates by treating each inserted coin, successively, as a bet on the three hands in the order listed, so a 30¢ bet would be distributed as 10¢ on each hand. The pay tables for each option, each of which pays a high-hand bonus when the maximum of 30 coins is bet, are shown in Table 8.14 on page 306.

Example 8.7. Suppose that the initial dealt hand is A♣ J♣ 9♣ 9♢ 5♡, with 30¢ bet per hand. This hand returns 30¢. If you discard 3 cards and draw to the pair of 9s, receiving the T♣, 7♠, and 7♡, the hand is now two pairs and pays 30¢. Two more cards are dealt, the 6♡ and 2♡, and the hand remains 2 pairs, which is a losing 7-card hand. The total loss on the 3-hand sequence is 30¢. ■

Since there is no player choice, the 5-card stud hand is easy to assess. Table 1.2 (page 22) gives the frequencies for the initial 5-card hands; all we need to do is separate out the various types of 4-of-a-kind hands and paying 1-pair hands. The 624 four of a kinds are distributed as follows:

- Aces with a 2, 3, or 4: 12.
- 2s–4s with an ace, 2, 3, or 4: 36.
- Aces with a 5–K: 36.
- 2s through 4s with a 5–K: 108.
- 5s through kings: 432.

The 1,098,240 hands containing a pair are evenly distributed among the 13 ranks, so there are 337,920 hands with a pair of jacks through aces and 675,840 hands of a pair of 3s through 10s. Combining these hand frequencies with the pay table shows that the 5-card stud game has a return percentage of 96.73% when betting 1 coin. The bonus payoff for a royal flush yields a 97.04% return rate when betting 30 coins per hand.

When the game moves to 5-card draw with Double Double Bonus poker rules, the return percentage can be computed using an online video poker analyzer such as the one at wizardofodds.com [136]. Software such as this uses an optimized algorithm to cycle through 3,986,646,103,440 different poker hands, representing every possible way to play out a dealt 5-card hand by discarding from 0–5 cards. The 5-card draw table in Table 8.14 yields a return rate of 99.56%.

For the 7-card stud hand, we can approximate the return percentage by using the recorded frequencies for 5-card hands among 7 cards (Table 2.2, page 64); this is an approximation because all possible final 7-card hands are not equally likely due to the 5-card draw option. The calculations are also slightly more complicated because of the enhanced payoffs for certain 7-card hands. However, one thing is clear: the pay table in use for the 7-card stud

TABLE 8.14: Pay tables for 3-Way Action Poker.

Hand	**1 coin**	**30 coins**
5-card stud		
Royal flush	2000	**120,000**
Straight flush	500	15,000
4OAK: Aces w/ 2,3,4	1000	30,000
4OAK: 2s–4s w/ A–4	400	12,000
4OAK: Aces	400	12,000
4OAK: 2s–4s	200	6000
4OAK: 5s–Kings	100	3000
Full house	25	750
Flush	15	450
Straight	9	270
3 of a kind	5	150
2 pairs	4	120
Pair, jacks or better	2	60
Pair, 3s–10s	1	30
5-card draw		
Royal flush	250	**24,000**
Straight flush	100	3000
4OAK: Aces + 2,3,4	400	12,000
4OAK: 2s–4s + A–4	160	4800
4OAK: Aces	160	4800
4OAK: 2s–4s	80	2400
4OAK: 5s–Kings	50	1500
Full house	9	270
Flush	6	180
Straight	4	120
3 of a kind	3	90
2 pairs	1	30
Pair, jacks or better	1	30
7-card stud		
4 aces + 3 2s–4s	2000	**120,000**
4 2s–4s + 3 As–4s	2000	60,000
7-card straight flush	800	24,000
4OAK + 3OAK	400	12,000
Royal flush	200	6000
Straight flush	25	750
4OAK: Aces	80	2400
4OAK: 2s–4s	40	1200
4OAK: 5s–Kings	20	600
Full house	5	150
Flush	3	90
Straight	2	60
3 of a kind	1	30

TABLE 8.15: 3-Way Action Poker: 7-card hands with enhanced payoffs.

7-card hand	**Count**
4 aces + 3 2s–4s	12
4 2s–4s + 3 As–4s	36
7-card straight flush	32
4OAK + 3OAK	576

game means that 113,255,660 seven-card hands, over 84.6% of the total, are non-winners. This simultaneously explains why the return rate of this third game under that assumption is only 42.63% with one coin wagered as well as pointing up the importance of good 5-card draw strategy in improving the eventual 7-card stud hand.

The 7-card hands with bonus payoffs are tabulated in Table 8.15. These 4 hands are so rare—there are $\binom{52}{7} = 133,784,560$ possible 7-card hands—that, despite their high payoffs of thousands of coins, they collectively contribute only about 1/400 of a bet to the expected return of the 7-card stud hand, whether the wager is for 1 coin or 30.

8.3 Optimal Strategy

Video poker is an exception to the rule that casinos make money from poker by extracting a small portion of each pot. The rules for a given machine, including its pay table, are set to guarantee that the casino retains a certain percentage of the total money wagered in the long run. This assumes that a player is using the optimal strategy appropriate to a given game; any deviation from the ideal strategy works to increase the casino's hold. This is the same principle that underlies setting the payoffs for casino table games: the house will get its predictable share of the money wagered in the long run because of the inherent design of the game and its wagers.

Though video poker is often classified with slot machines due to their physical resemblance, one key difference between them is that slot algorithms are closely-guarded trade secrets. By Nevada law, the logic underlying video poker programming must, by contrast, give an equal chance of drawing any card still remaining in the deck at any time. This big-picture predictability, which requires no knowledge of the specifics of the algorithm that matches random numbers to playing cards, makes it possible to develop meaningful strategies that give the best possible long-term return to the player. Many other states with casinos have based their regulations on Nevada's, so the same quality assurance provided in America's first state with legal casinos is available elsewhere.

Unlike live poker, video poker is a game of perfect information. There are no hidden cards or issues of opponent psychology to consider in playing a hand—just pure probability to guide a player toward the best possible decision. Since there are no human opponents whose thinking might be manipulated by occasionally changing up your actions, there is never any value in holding a kicker when drawing to a pair in Jacks or Better video poker, as we examined on page 85. Your chance of improving your hand is greater when you draw 3 cards to a pair instead of drawing 2 cards to a pair plus a kicker, and this is the only mathematical consideration that should guide your action. As we saw on page 85 when examining draw poker, the only advantage when drawing 2 cards to a pair and a kicker instead of drawing 3 to a pair is a slightly greater chance of making 2 pairs, but since aces up pays the same as any other 2-pair hand, this strategy does not increase the video poker payoff.

Until the 1981 publication of a paper on optimal "electronic poker" strategy by David Sklansky, video poker was on several occasions classified as a game of chance rather than skill [139]. Given a video poker game and its accompanying pay table, it is possible to devise an optimal strategy for the game. This strategy advises the player on how a given dealt hand should be played, based on the expected return of the hand. The ideal strategy for a given machine and pay table has been determined through careful calculation combined with computer simulations. It is important to identify the best strategy for the specific game being played, though.

Example 8.8. In Deuces Wild video poker, if your original hand contains 2 pairs and no deuces, correct strategy calls for discarding one pair—it doesn't matter which—and drawing 3 cards, since the lowest winning hand is 3 of a kind and a 3-card draw increases the chance of reaching the pay table.

In Jacks or Better, a 2-pair hand should draw only 1 card; it's already a winning hand and has a small chance to improve to a full house. ■

Jacks or Better

Video poker strategies are typically presented as a table, with hands listed in decreasing order of their value. For a Jacks or Better game with Table 8.2 as pay table, the strategy in Table 8.16 leads to the 99.54% payback percentage mentioned on page 288. The corresponding 9/5 machine's payback percentage is 98.8%, and an 8/5 machine that is otherwise the same pays back at 97.6% [84].

Table 8.16 should be read from the top down until you reach a combination contained in the hand you hold, and then you should discard appropriately and draw the number of cards indicated. High cards, for the purpose of Jacks or Better video poker, are the jack, queen, king, and ace—those cards that, when paired, pay off.

TABLE 8.16: Video poker strategy for standard 9/6 Jacks or Better [84].

	When dealt	**Draw**
1:	Royal flush	0
2:	Straight flush	0
3:	4 of a kind	0
4:	4-card royal flush	1
5:	Full house	0
6:	Flush	0
7:	3 of a kind	2
8:	Straight	0
9:	2 pairs	1
10:	4-card straight flush	1
11:	High pair: J, Q, K, or A	3
12:	3-card royal flush	2
13:	4-card flush	1
14:	Low pair, 2–T	3
15:	4-card outside straight	1
16:	3-card straight flush	2
17:	Suited JQ, JK, or QK	3
18:	4-card inside straight (3–4 high cards)	1
19:	Suited JA, QA, or KA	3
20:	Nonsuited JQK	2
21:	Suited JT, QT, or KT	3
22:	1 or 2 high cards	3–4
23:	5 mixed low cards (a razgu)	5

On lines 4 and 12 of Table 8.16, the pursuit of a royal flush is prioritized over lesser payoffs. Occasionally, perfect strategy may call on the gambler to break up a dealt winning combination in order to draw cards toward completing a royal flush. Unlikely as successfully completing a royal by drawing one or two cards may be, the high payoff at max coins leads to an expected value in excess of value of the hand being broken up.

Example 8.9. If you are dealt

$$9\diamondsuit\ T\diamondsuit\ J\diamondsuit\ K\diamondsuit\ A\diamondsuit,$$

perfect Jacks or Better strategy calls for breaking up the flush, discarding the $9\diamondsuit$, and drawing one card in hopes of pulling the $Q\diamondsuit$ needed to complete the royal flush. This is the best choice because Line 4 outranks Line 6 in Table 8.16. The greatly enhanced payoff for a royal flush with max coins wagered makes this the better play, as we can see by computing expected values. If you hold the $9\diamondsuit$ and collect on the flush, your expected profit is \$25 when betting

TABLE 8.17: Possible outcomes when drawing 1 card to suited $TJKA$.

Result	Net Payoff	Probability
Royal flush	4695	1/47
Flush	25	7/47
Straight	15	3/47
Pair (J, K, or A)	0	9/47
Nonpaying hand	–5	27/47

5 coins, assuming a $1 coin. If you discard the 9, the outcomes in Table 8.17 are possible.

The expected value of the hand if the 9♢ is discarded is

$$E = (4695) \cdot \left(\frac{1}{47}\right) + (25) \cdot \left(\frac{7}{47}\right) + (15) \cdot \left(\frac{3}{47}\right) + (0) \cdot \left(\frac{9}{47}\right) + (-5) \cdot \left(\frac{27}{47}\right)$$
$$= \frac{4780}{47} \approx \$101.70.$$

It follows that, even though the probability of winding up with nothing after the draw is over 50%, your expected return is over four times greater by giving up on the sure $25 payoff for the chance of hitting a royal flush. ■

The pursuit of royal flushes adds a level of volatility to video poker. A player's bankroll may go through wild swings in a lifetime of video poker, generally trending downward until hitting a royal flush and soaring upward, then slowly dropping again until the next royal, over and over.

Another consequence of following optimal video poker strategy is that straight flushes are rarer than in standard poker games. The ratio of straight flushes to royal flushes in standard poker is 9 to 1. Since the best way to play certain video poker hands is to discard certain 2-card straight flushes in favor of holding a single face card or ace, for example when dealt

$$K\diamondsuit\ 8\heartsuit\ 7\heartsuit\ 3\clubsuit\ 2\spadesuit,$$

a video poker player will draw into fewer straight flushes, which are only 5.5 times more numerous than royals [20].

Some viable-looking poker hands have little or no value in Jacks or Better video poker. A 4-card inside straight with no high cards, 7♢ 6♣ 4♣ 3♠ for example, can only improve to a winning hand if a 5 is drawn. The expected return on this hand is –0.66 coins, which is less than the –0.64-coin return on a razgu [53]. Given our understanding that inside straights are typically not very valuable in draw poker, the mathematical conclusion might not be surprising: the optimal play is to draw 5 new cards.

Joker Poker

The optimal strategy for Joker Poker with a pay table given by Table 8.7 is shown in Table 8.18. What to do with a given hand depends on whether or

TABLE 8.18: Optimal strategy for Joker Poker [84].

Hands without the joker		**Hands with the joker**	
Holding	**Draw**	**Holding**	**Draw**
Royal flush	0	5 of a kind	0
Straight flush	0	Royal flush	0
4-card royal flush	1	Straight flush	0
4 of a kind	1	4 of a kind	1
Full house	0	Full house	0
Flush	0	Flush	0
Straight	0	4-card royal flush	1
4-card straight flush	1	4-card straight flush	1
3 of a kind	2	Straight	0
2 pairs	1	3 of a kind	2
3-card royal flush	2	3-card straight flush	2
4-card flush	1	4-card flush	1
4-card outside straight	1	4-card straight	1
3-card straight flush	2	3-card outside straight	2
Pair	3	Joker	1
4-card inside straight	1		
3-card flush	2		
2-card royal flush	3		
3-card outside straight	2		
2-card straight flush	3		
Razgu	5		

not the joker is present, and the number of cards to be drawn takes the joker into account. For example, a player holding 4 of a kind should always draw 1 card instead of 0, because the joker makes it possible to improve 4 of a kind to 5 of a kind. Since any hand with the joker is at least a pair, there is no razgu line on the joker side of the table, but since 3 of a kind is the lowest paying hand, there's no value attached to a pair including the joker, and a player holding the joker with 4 unmatched cards is advised to hold only the joker and draw 4 new cards.

In examining Table 8.18, we note that there are some differences in optimal strategy with and without the joker. If you do not hold the joker, you should not break up a dealt straight to pursue a straight flush. For example, in the hand $J♣\ T♡\ 9♡\ 8♡\ 7♡$, you should cash that hand for 5 coins rather than discard the $J♣$ in the hope of filling in the straight flush. There are 4 possibilities if you discard:

- **Straight flush**. 3 cards will complete the straight flush: the joker, $J\heartsuit$, and 6♡. The probability of a straight flush and its 50-coin payoff is $\frac{3}{48}$.
- **Flush**. Any of the 7 hearts not listed so far will turn this hand into a flush paying 6 coins. The probability is $\frac{7}{48}$.
- **Straight**. The straight originally held is restored if you draw any other 6 or jack, and 5 of those remain. $P(\text{Straight}) = \frac{5}{48}$.
- **Losing hand**. Any of the remaining 33 cards create a final hand which loses 1 coin. The chance of giving up the straight and losing is $\frac{33}{48}$.

The expected value of this hand when discarding and drawing is then

$$E = (50) \cdot \frac{3}{48} + (6) \cdot \frac{7}{48} + (5) \cdot \frac{5}{48} + (-1) \cdot \frac{33}{48} = \frac{186}{48} < 5,$$

indicating that holding the straight is the better choice. This expectation is also less than 6, indicating that a dealt flush including 4 cards to a straight flush should also not be broken up—as Table 8.18 indicates.

This assumes a straight flush open at both ends. If you are dealt a straight containing an inside straight flush like $T\heartsuit$ 9♡ 8♣ 7♡ 6♡, discarding the 8♣ and going for a straight flush has $E = \frac{128}{48}$, which is again less than 5. Holding the straight is still favored.

If you are dealt a hand such as 9♢ 8♢ 7♢ W 5♣, including the joker, the two hands "Straight" and "4-card straight flush" switch places. Discarding the 5♣ and drawing 1 card is favored. The probability of a 1-card draw giving a straight flush rises to $\frac{4}{48}$, since the 5, 6, 10, or jack of diamonds will all complete a straight flush. The expected value of this hand is $E = \frac{266}{48} > 5$.

Deuces Wild

Table 8.19 on page 313 gives the optimal strategy for a full-pay Deuces Wild video poker game returning 100.76%, whose pay table is shown in Table 8.9. The table is sorted by the number of deuces that the player holds; it is of course reasonable that this strategy never calls for discarding a wild deuce.

Five Deck Frenzy

Five Deck Frenzy is a variation on Five Deck Poker that offers a progressive jackpot (page 147) if a player hits 5 aces of spades [84]. The jackpot starts at $200,000 and has the potential to grow to truly astonishing levels because the triggering hand is so rare. The pay table for Five Deck Frenzy is shown in Table 8.20.

TABLE 8.19: Optimal strategy for full-pay Deuces Wild [131].

Holding	**Draw**
4 deuces	
4 deuces	0
3 deuces	
Wild royal flush	0
5 of a kind	0
3 deuces	2
2 deuces	
4 of a kind or better	0
4-card wild royal flush	1
4-card open straight flush, 67–9T	0
2 deuces	3
1 deuce	
4 of a kind or better	0
4-card wild royal flush	1
Full house	0
4-card open straight flush, 567–9TJ	0
3 of a kind or better	2
4-card straight flush	1
3-card wild royal flush, king- or queen-high or JT	2
3-card open straight flush	2
1 deuce	4
0 deuces	
Royal flush	0
4-card royal flush	1
3 of a kind or better	0–2
4-card open straight flush: 4567–9TJQ	1
Suited QJT	2
4-card inside straight flush with 1 gap: 3456, 3457–9TJQ, or A345	1
3-card royal flush: AKQ, AKJ, AQJ, KQJ	2
Any pair (discard one pair if dealt 2 pairs)	3
4-card flush	1
4-card open straight: 4567–$TJQK$	1
3-card open straight flush: 345–9TJ: 2	
Suited JT	3
3-card inside straight flush: 346–356, 347–69T, 78J–79J, 89Q–9QK	2
4-card inside straight, 8-low or better	1
Suited QJ, JT	3
4-card inside straight: 3456–79TJ	1
Razgu	5

TABLE 8.20: Five Deck Frenzy pay table [52].

	Coins wagered	
Poker hand	**1**	**5**
5 $A\spadesuit$s	20,000	Jackpot
Suited 5 of a kind	1000	10,000
Royal flush	250	4000
Unsuited 5 of a kind	55	275
Straight flush	50	250
Suited 4 of a kind	20	100
Suited full house	12	60
Unsuited 4 of a kind	10	50
Unsuited Full house	6	30
Flush	4	20
Suited 3 of a kind	4	20
Straight	3	15
Unsuited 3 of a kind	2	10
Suited 2 pairs	2	10
2 pairs	1	5

Consider an unwise strategy where a gambler pursues only the progressive jackpot, discarding every card that is not the $A\spadesuit$ and drawing. The probability of an initial hand with k aces of spades, $0 \leqslant k \leqslant 5$, is

$$P(k) = \binom{5}{k} \cdot \left(\frac{1}{52}\right)^k \cdot \left(\frac{51}{52}\right)^{5-k},$$

and the probability of filling in $5-k$ slots with the $A\spadesuit$ on the draw is

$$Q(5-k) = \left(\frac{1}{51}\right)^{5-k}.$$

It follows that the probability of drawing the jackpot hand using this strategy is

$$\sum_{k=0}^{5} [P(k) \cdot Q(5-k)] = \sum_{k=0}^{5} \binom{5}{k} \left(\frac{1}{52}\right)^5 = \frac{32}{380,204,032} = \frac{1}{11,881,376}.$$

This may be regarded as an upper bound for the probability of 5 aces of spades in actual gameplay, since an optimal strategy for Five Deck Frenzy would favor keeping some hands such as a royal flush or 5 $K\diamondsuit$s over the single-minded pursuit of the progressive jackpot. The long-term probability of 5 spade aces, using optimal Five Deck Frenzy strategy, is somewhat lower, 1 in 14,896,150 [52].

What is this optimal strategy? Certainly one would expect to hold $A\spadesuit$s more often than in Jacks or Better or similar games, but how does the progressive jackpot influence the best option for players?

There are 38 hand types enumerated in Five Deck Frenzy optimal strategy, which is shown in Table 8.21 on page 316. A singleton $A\spadesuit$ ranks 35th, ahead of only the razgu and certain 2-card inside straight flushes. Recall that an *inside* straight flush consists of 4 suited cards in sequence with a 1-card gap, such as $\heartsuit$3457.

- A *2-card inside* straight flush consists of 2 suited cards with a single gap in between them; for example, $J\spadesuit\ 9\spadesuit\ Q\diamondsuit\ 6\heartsuit\ 4\diamondsuit$.
- A *2-card double inside* straight flush contains 2 suited cards with a 2-rank gap, such as $J\spadesuit\ 8\spadesuit\ Q\diamondsuit\ 6\heartsuit\ 4\diamondsuit$.
- A *3-card double inside* straight flush contains 3 cards, suited and in sequence but for 2 gaps, for example, $\clubsuit 68T$ or $\heartsuit$367.

The highest-ranked hand where a gambler should discard an $A\spadesuit$ is on line 6: a suited 4 of a kind. When holding a hand like $T\spadesuit\ T\spadesuit\ T\spadesuit\ T\spadesuit\ A\spadesuit$, the 20 for 1 payoff on the hand as it stands, with a $\frac{1}{51}$ chance of improving to a suited 5 of a kind and its 1000 for 1 payoff, gives a better result than the longshot of drawing to a singleton $A\spadesuit$.

Example 8.10. When holding 4 $A\spadesuit$s and drawing 1 card, 3 outcomes are possible:

- *The hand fails to improve.* Assuming that the discard is not an ace of another suit—and that hand should be broken up, since 4 $A\spadesuit$s, on line 2 of Table 8.21, outranks an unsuited 5 of a kind at line 7—the probability of no improvement is $\frac{47}{51}$, since only another ace will lift this hand past a suited 4 of a kind. The payoff on a suited 4 of a kind is 20 for 1.
- *The hand draws a non-spade ace and counts as an unsuited 5 of a kind.* The probability of 5 unsuited aces after the draw is $\frac{3}{51}$, and the payoff is 55 for 1.
- *The draw produces the fifth $A\spadesuit$.* This 1 in 51 shot collects the progressive jackpot if the gambler has wagered 5 coins.

Five Deck Frenzy was a quarter game, so 1 coin = 25¢. The expected return on this 1-card draw with the minimum jackpot of \$200,000 (800,000 coins) and max coins wagered is then

$$E = (100) \cdot \frac{47}{51} + (275) \cdot \frac{3}{51} + (800,000) \cdot \frac{1}{51} - 5 \approx \$3947.40.$$

■

TABLE 8.21: Five Deck Frenzy: Optimal strategy [52].

	Initial hand	**Draw**
1:	5 $A♠$	0
2:	4 $A♠$	1
3:	Suited 5 of a kind	0
4:	Royal flush	0
5:	3 $A♠$	2
6:	Suited 4 of a kind	1
7:	Unsuited 5 of a kind	0
8:	Straight flush	0
9:	4-card royal flush	1
10:	4 of a kind	1
11:	Suited full house	0
12:	Suited 3 of a kind	2
13:	Unsuited full house	0
14:	Flush	0
15:	3 of a kind	2
16:	4-card straight flush	1
17:	Straight	0
18:	2 suited pairs	1
19:	2 $A♠$	3
20:	4-card inside straight flush	1
21:	2 unsuited pairs	1
22:	Suited TJQ	2
23:	Suited pair	3
24:	Suited JQK	2
25:	4-card flush	1
26:	3-card royal flush, ace high	2
27:	One pair	3
28:	3-card straight flush	2
29:	4-card straight	1
30:	3-card inside straight flush	2
31:	3-card double inside straight flush	2
32:	3-card flush	2
33:	2-card royal flush	3
34:	2-card straight flush: 34 through 9T	3
35:	1 $A♠$	4
36:	2-card inside straight flush	3
37:	2-card double inside straight flush	3
38:	Razgu	5

As the progressive jackpot increases, the relative ranking of a hand containing a single $A♠$ can change relative to the other hands in Table 8.21. Specifically, two-card royal flushes (line 33) are inferior to an $A♠$ as the jackpot grows. Table 8.22 shows how large the jackpot must be to favor holding a lone $A♠$ over certain 2-card royals and 3-card flushes (line 32).

TABLE 8.22: Five Deck Frenzy: Strategy changes with increasing jackpot [52].

Jackpot	**Hold $A♠$ over**
\$238,000	Suited AK
\$290,000	Suited KQ
\$340,000	Suited QJ
\$400,000	Suited JT
\$660,000	3-card flush

Double Down Stud

Since there is no discard-and-draw choice to be made in Double Down Stud, the only strategy choice lies in the decision whether or not to double the initial bet. This decision should be based on the probability of achieving any given hand in Double Down Stud, which is easily calculated since there is only 1 card left to be revealed when the choice must be made.

One strategy is self-evident: double your bet if you have a paying hand in four cards, as with $Q♣$ $9♠$ $Q♡$ $7♡$ from Example 8.6, before the fifth card is turned up. You're guaranteed a win, so you might as well double your payoff with absolutely no increased risk. What is the probability that a 4-card hand is already a winner?

The number of 4-card winners can be counted using a variation of our master formula for counting different hand types. The following 4-card hands are winners before the fifth card is dealt:

- 4 of a kind: There are 13 4-card four-of-a-kind hands.
- 3 of a kind: These number
$$13 \cdot \binom{4}{3} \cdot 48 = 2496.$$
- Two pairs: This 4-card hand can be drawn in
$$\binom{13}{2} \cdot \binom{4}{2}^2 = 2808$$
ways.

- One pair, 6s or higher: There are 9 ranks from which to choose the pair, and
$$9 \cdot \binom{4}{2} \cdot \binom{12}{2} \cdot \binom{4}{1}^2 = 57,204$$
possible winning hands.

The probability of a 4-card winner is then found by summing the numbers above and dividing by $\binom{52}{4} = 270,725$; we have

$$P(\text{4-card winner}) = \frac{62,521}{270,725} \approx .2309;$$

just under 1 in 4 hands will be a winner before the fifth card is dealt.

This leaves over 75% of all hands where a choice must be made based on incomplete information. When should you double your bet in the hope that the final community card will turn your non-winning four-card hand into a winner? Careful consideration of the probabilities shows that only the following non-winning four-card hands justify doubling your bet [125]:

- A four-card flush, including straight and royal flushes.

  Suppose that you hold the 4-card flush $\diamondsuit AJ87$, which admits no straight flush draw. The fifth card can complete a flush, or can raise your hand to a pair. Pairing any card will create a winning hand, although a pair of 7s or 8s merely pushes.

  The probability of a flush is $\frac{9}{48}$, the probability of a pair paying even money (jacks or aces) is $\frac{6}{48}$, and the probability of pushing with a pair of 7s or 8s is also $\frac{6}{48}$. The expected value of this hand when doubled is then
$$E = (16) \cdot \frac{9}{48} + (2) \cdot \frac{6}{48} + (0) \cdot \frac{6}{48} + (-2) \cdot \frac{27}{48} = \frac{102}{48} > 0,$$
  justifying the raise.

  A 4-card flush with the possibility of drawing into a straight flush with the 5th card has an even higher expected value, and so is also worth the increased risk of a doubled bet.

- A four-card open-ended straight, but not an inside straight.

  Suppose that you hold the 5♡ and the first three community cards are 3♢ 4♣ 6♡, giving you a draw to an open-ended straight. Whether you double your bet or not, there are eight cards—four 2s and four 7s—that will complete the straight. In addition, if the fifth card is a 6, you will push on your hand with a pair of 6s. The expectation if you do not double your bet is

$$E = (5) \cdot \frac{8}{48} + (0) \cdot \frac{3}{48} + (-1) \cdot \frac{37}{48} = \$\frac{3}{48} = \$.0625,$$

a positive value which means that you currently hold an advantage—so this hand is worth doubling. If you double the bet, the expectation also doubles, to \$.125. Your advantage remains 6.25% either way.

Note that the presence of the 6 in your hand makes a difference here. If you hold an unsuited 2345, its expected value is zero, so not doubling and doubling both mean playing a fair game. If a game is fair, the only point in playing is any possible enjoyment you might receive from the excitement of the action.

As noted above, an inside straight without a flush draw is a different matter. If you are holding 4♠ 5♢ 7♠ 8♠, your expectation is

$$E = (5) \cdot \frac{4}{48} + (0) \cdot \frac{6}{48} + (-1) \cdot \frac{38}{48} = -\$\frac{18}{48} = -\$0.375$$

—a negative value that only gets more negative if you double your bet. Taking back your bet is not an option, so the best strategy is not to double your bet and hope that the fifth card is one of the ten (four 6s, three 7s, and three 8s) that allows you to avoid a loss.

- An unsuited $JQKA$ hand. This is the only closed-end straight (a straight involving an ace, which can only be completed at one end) that should be doubled.

  This hand is included because of its considerable possibility of leading to a paying pair on the fifth card. The expectation for this hand is

  $$E = (5) \cdot \frac{4}{48} + (1) \cdot \frac{12}{48} + (-1) \cdot \frac{32}{48} = \$0,$$

  so the double is justified since the expectation is not negative. At this point, you're playing a fair game, and as with the 2345 hand analyzed above, doubling may be the correct short-term decision as you play for a bigger immediate gain against the best odds you're likely to find in the casino.

Finding the house advantage for a single hand of Double Down Stud requires careful attention to the effects of a bet-doubling strategy. Blindly doubling every wager gives a HA of 27.4%—too much money is lost doubling hands with little or no probability of winning. At the other extreme, if you never double your bet, you face the same 27.4% HA, because you lose higher payoffs on guaranteed winning hands [9]. The challenge in computing the theoretical house edge lies in separating out when a winning hand such as K♣ K♡ 9♠ 3♣ 2♡ arises from a doubled bet, when the facedown card is not a king, and when it comes with a non-doubled bet, as when one of the kings is the final card revealed.

The following hands always come from a doubled bet if the strategy detailed above is followed:

- Royal flush
- Straight flush
- Four of a kind
- Full house
- Flush

This covers only 9516 of the 2,598,960 possible hands. All other final poker hands could result from doubling a wager, or not.

Example 8.11. Consider three-of-a-kind hands. There are 54,912 of these, 4224 of each rank. If the triple is of 6s through aces, at least two of them will appear in the four exposed cards, and so a player following the recommended strategy will always double his or her bet. For low triples of 2s through 5s, the hand will only be doubled if the triple occurs among the first four exposed cards; that is, if the hole card is *not* one of the triple.

The probability that a "low" three-of-a-kind hand will not be doubled is simply the chance of the hole card completing the triple: $\frac{3}{5}$—hence, $\frac{2}{5}$ of all low three of a kinds will be doubled. Combining these two cases shows that the probability of a three of a kind being doubled is

$$\frac{9 \cdot 4224 + 4 \cdot \frac{2}{5} \cdot 4224}{54,912} = \frac{44,774.4}{54,912} \approx .8154,$$

and so we may say that a typical winning three-of-a-kind hand carries a wager of 1.8154 betting units. ■

Example 8.12. Two-pair hands can be analyzed similarly. There are 123,552 hands with two pairs, which can be sorted into 78 subsets of 1584 hands each based on the paired cards. These subsets can be broken down by the ranks of the pairs:

- If both pairs are "high" cards—6 through ace—then at least one such pair will be among the four known cards, and so these hands will always be doubled. There are $\frac{9 \cdot 8}{2} = 36$ such combinations of high pairs.
- Two-pair hands where both pairs are "low" cards—2 through 5—comprise only 6 of these 78 subsets: 5-4, 5-3, 5-2, 4-3, 4-2, and 3-2. These hands will be doubled if the odd card is the hole card, an event which occurs with probability $\frac{1}{5}$.

- "Mixed" hands, with one high and one low pair, will be doubled provided that the high pair falls among the four exposed cards, or alternately, if the hole card is the odd card or one of the low pair. There are 36 subsets of two-pair hands which contain a mixed pair, and the probability that they will be doubled is $\frac{3}{5}$.

The total probability of two pairs being doubled is then

$$\frac{36 \cdot 1584 + 6 \cdot \frac{1}{5} \cdot 1584 + 36 \cdot \frac{3}{5} \cdot 1584}{123,552} = \frac{93,129.2}{123,552} \approx .7538,$$

giving a weight of 1.7538 betting units to the average two-pair hand. ■

Continuing in this fashion through the remaining poker hands yields a house advantage for Double Down Stud of 2.67%, or a payback percentage of 97.33% [125].

8.4 Additional Game Options

Multiple Hand Play

Many video poker machines offer the option of multiple draws—3, 5, 10, or as many as 100—to a single starting hand. One hand is dealt, and after the player chooses which cards to hold and which to discard, the held cards are also held in every other hand. Each hand is then played out separately with its own draw from the 47 cards left in the deck.

Example 8.13. In the 3-hand Jacks or Better game shown in Figure 8.2, the $K\spadesuit$ has been held in the first hand and copied into each of the other hands.

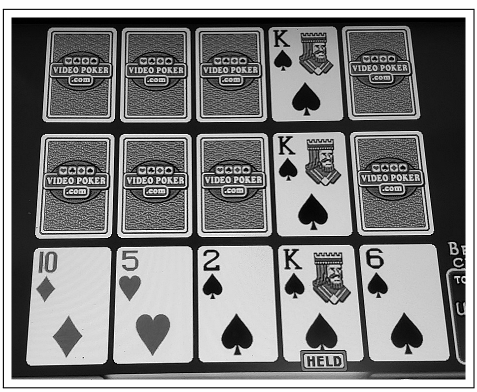

FIGURE 8.2: Triple Play video poker with a single card held across all 3 hands.

The player now has 3 chances to draw 4 cards and improve a 1-king hand to something higher. ■

Each hand in a multiple-hand game requires a separate wager. This increases the initial investment, but when a dealt pat hand generates a screen full of winners, the payoff can be immense.

Example 8.14. In Figure 8.3, a dealt hand containing 2 pairs is copied across all 3 hands, which guarantees that all 3 final hands will be winners.

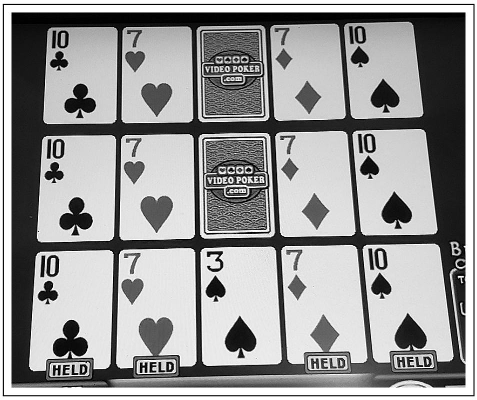

FIGURE 8.3: Triple Play video poker: Two pairs are held in all 3 hands.

When the Draw button was pressed, the hands filled in to 2 full houses and a 2-pair hand, as shown in Figure 8.4.

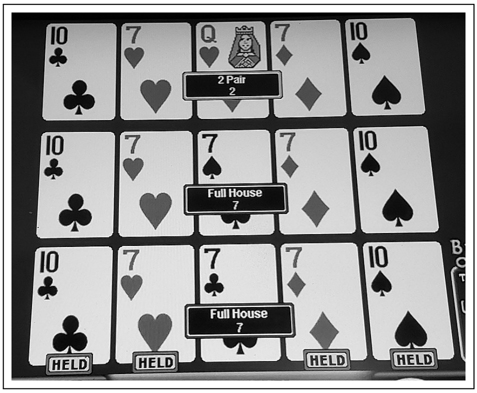

FIGURE 8.4: Three completed hands starting with 2 pairs in Triple Play video poker.

An initial wager of 3 coins led to a 16-coin payoff. ■

Multiple-hand games require no change in strategy. They should be played using the same strategy table as a single-hand version of the same game. The probability of winning is constant across each of the hands; what changes is the variance on the total bet. Wins and losses can be bigger, and the player's bankroll will fluctuate more as a result.

In a 50-hand game where the player draws 1 card to $K\spadesuit\ K\heartsuit\ 3\spadesuit\ 3\heartsuit$, the number of hands that successfully draw into a full house is a binomial random variable X with parameters $n = 50$ and $p = \frac{4}{47}$. The expected number of full houses is $np \approx 4.26$, though the probability of drawing no full houses is

$$P(X = 0) = \binom{50}{0}\left(\frac{4}{47}\right)^0 \cdot \left(\frac{43}{47}\right)^{50} \approx .0117,$$

meaning that approximately 1 time in 85, no hands will draw the necessary third 3 or king.

This is, of course, the worst-case scenario. In a cheerier calculation, the probability of drawing 9 or more full houses in 50 hands is about .0239–more than twice the chance of drawing none.

Progressive Jackpots

In addition to progressive jackpots on nonstandard video poker games like Five Deck Frenzy, some installations of games like Jacks or Better have reconfigured pay tables that include a progressive payoff for a royal flush. These jackpots may be shared among a bank of machines with the jackpot awarded to the player on the first machine that hits a royal. A *1% meter* progressive video poker machine adds 1% of each wager to the progressive jackpot until it is won.

That 1% given out to jackpot winners must be recouped somewhere. Progressive video poker and slot machines typically pay less on smaller payoffs in order to preserve an overall casino advantage, as when a 9/6 machine is reprogrammed as an 8/5 machine and the hold percentage rises from .46% to 2.70% [144]. This permits raising the effective payoff for the progressive jackpot while the casino can still make money—up to a point.

If the jackpot grows past a certain size determined by the pay table, the machine's return to players may exceed 100%, and the advantage will then lie with the knowledgeable gambler. To take advantage of these rare circumstances is challenging, though, because the event triggering the progressive jackpot is by definition rare, and it is quite likely that multiple experienced players will be drawn to a game with a positive expectation, increasing the difficulty of being the first player to hit a royal and capture the jackpot.

Pursuing a progressive jackpot may call for changing one's strategy. This occurs when the jackpot is sufficiently large that the expected value of the hand when drawing to a royal flush exceeds the hand's expectation when following the optimal strategy table.

Example 8.15. Consider an 8/5 Jacks or Better machine with a progressive jackpot on royals. Optimal strategy for an 8/5 game calls for breaking up dealt straights and flushes that contain 4 cards to a royal flush, but the straight flush $KQJT9$ takes precedence over a 4-card royal. With a sufficiently high jackpot J, however, breaking up this straight flush to pursue the enhanced payoff for a royal has a higher expectation.

A straight flush has a return of 250 coins for a 5-coin wager, but if you are dealt the straight flush $\diamondsuit KQJT9$, discarding the 9$\diamondsuit$ leaves a 1-card draw to a royal with the possible outcomes shown in Table 8.23.

TABLE 8.23: 8/6 Jacks or Better progressive: Outcomes when betting 5 coins and drawing 1 card to $\diamondsuit KQJT$ after discarding the 9$\diamondsuit$.

Outcome	**Probability**	**Payoff**
Royal flush	$1/47$	Jackpot J
Flush	$7/47$	25
Straight	$6/47$	20
High pair	$9/47$	5
Nothing	$24/47$	0

The expected value in this setting is

$$E = \frac{J + 25(7) + 20(6) + 5(9)}{47} = \frac{J + 340}{47}.$$

This exceeds the \$250 value of a straight flush if $J > 11,410$ coins—\$2852.50 on a quarter machine, or \$570.50 when playing for nickels—and so in those circumstances, discarding the sure thing and going for the royal has a higher expectation. ■

It is, of course, understandably tough to follow the math and give up a guaranteed 245-coin net win for an outcome that has over a 95% chance of being lower, and over a 50% chance of a net 5-coin loss. Nonetheless, the long-term prospects are better if you do. However, since the probability of a dealt king-high straight flush, where this change to optimal strategy comes into play, is the same $1/649,740$ as the chance of a dealt royal flush, you could be forgiven for holding all 5 cards, taking the 250-coin payoff, and cashing out. Unless you take up playing video poker for a living, it could be more than a lifetime before you could expect to be in this situation.

Sequential Royals

In Example 1.15, we considered a million-dollar sequential royal flush (SRF) promotion. Some video poker machines have added sequential royals ($AKQJT$ or $TJQKA$, in order) as the highest-paying hand. If there is no significant reduction in the payoff for a non-sequential royal, then the expected return on the game increases. If this is coupled with a progressive jackpot awarded to the holder of a sequential royal, it is possible that the game may pay off more than 100%—again assuming perfect play, which would have to include different strategy to incorporate the pursuit of certain sequential royals. As a general guideline, altering video poker strategy to pursue sequential royals is only the superior play if you are dealt at least 2 cards of an SRF in the correct positions. Note that the queen must land in the center position in either order.

However, one must be careful to inspect the pay table for changes to the payoffs on more common hands. Table 8.24 shows the pay table for a 5¢ Bonus Poker video game including a sequential royal bonus at the Aliante Casino in North Las Vegas, Nevada together with the pay table for a nearby Bonus Poker machine without the sequential royal bonus.

TABLE 8.24: Aliante Casino Bonus Poker pay tables.

	SRF bonus		**No SRF bonus**	
Poker Hand	**1 coin**	**5 coins**	**1 coin**	**5 coins**
Sequential royal flush	250	Jackpot	250	4000
Royal flush	250	4000	250	4000
Straight flush	50	250	50	250
Four aces	80	400	80	400
Four 2s, 3s, or 4s	80	400	40	200
Four 5s through Ks	80	400	25	125
Full house	8	40	6	30
Flush	5	25	5	25
Straight	4	20	4	20
Three of a kind	3	15	3	15
Two pair	1	5	2	10
Pair of jacks or higher	1	5	1	5

The progressive jackpot at the time was $63,835.82, a handsome 255,341–1 payoff for a 25¢ bet.

With the SRF bonus, the four-of-a-kind hands are lumped together and paid at the same rate, the top rate of 80 for 1 paid in the base game. but the casino makes up for this largesse and funds the progressive jackpot by reducing the payoff on two pairs from 2 for 1 to a mere push. Players eschewing the SRF bonus take on the further burden of lower payoffs for full houses and some four of a kinds, but will make a 1-unit profit anytime they draw into 2 pairs.

The Max

At the Tropicana Casino in Atlantic City, *The Max* was a bonus opportunity present on six 25¢ machines, triggered when a player or players had hit four-of-a-kind hands when betting a total of 5 coins [158]. This could be one hand when betting 5 coins, or 5 hands betting a single coin, or any other assortment of winning wagers.

When the bonus was active, the next four of a kind paid 125 for 1 instead of the standard 25 for 1. For a 5-coin wager, this raised the payoff from $31.25 to $156.25. This apparent casino largesse was paid for by setting the game as an 8/6 Jacks or Better game modified to pay only even money on two pairs. This pay table yielded a payback percentage of 85.33%—nearly a 15% HA—against standard Jacks or Better optimal strategy under ordinary payoffs, but in Max mode, the payback leapt to 108.96%, giving players an edge of almost 9%.

Optimal strategy for The Max changed standard 8/6 JOB strategy to downplay 2-pair hands and pursue four of a kinds more aggressively when the machine was in Max mode. When holding two high pairs such as $AAJJ8$, the player should discard one pair—it doesn't matter which one—and draw 3 cards. Since two pairs pays the same 1 for 1 as a high pair, little is lost in going after the lucrative four-of-a-kind hand.

Max mode, once triggered, remained active until the next four-of-a-kind hand was hit. This meant that an advantage player could enjoy the 9% edge without playing through the 15% HA by watching a Max machine until another player entered into Max mode and left without triggering the bonus. A player who did this successfully could earn an estimated $55 per hour before hitting that last four of a kind and walking away with the bonus [158].

Random Bonuses

Video poker is indisputably a game of skill once the cards are dealt. Some game innovations have included additional payoffs that are based on luck, with an eye toward drawing in new players with the potential of bonus payoffs. Bonus options are typically added to video poker games with lower player return percentages, such as 8/5 Jacks or Better with its 97.3% payback. Since these bonus options are available to players without an additional wager, the base game needs to be able to absorb the increased payoff to players without exceeding a 100% return.

Alternately, the bonuses may be funded by changing the pay table completely, lowering the payoffs for completed hands. This tips the balance of a particular game a little more toward luck than skill, a practice that can be repellent to serious video poker players.

Lucky Deal

One of the first bonus opportunities to arrive on video poker machines was *Lucky Deal*, which debuted on 8/5 Jacks or Better machines [49]. Lucky Deal awarded double the usual payout on dealt hands of a straight or higher. Since the payoff was awarded before the opportunity to draw, this is based entirely on luck.

These winning hands are rare, of course. The probability of a Lucky Deal win, using Table 1.2, is

$$\frac{19,716}{2,598,960},$$

approximately .75%. The low-probability event kept the net player return below the 99.54% that 9/6 Jacks or Better video poker offers players.

Blackjack Power

Blackjack Power is a bonus feature that was added to some video poker games in the 1990s [49]. This bonus was paid on the initial dealt hand, and so pursuing it did not require any change in strategy. Blackjack Power machines paid a 1 for 1 bonus if the first 5 cards contained a suited black ace and jack: either $A\spadesuit$ $J\spadesuit$ or $A\clubsuit$ $J\clubsuit$. If all 4 cards appeared in the hand, the payoff was 50 for 1. The "for 1" payoff had the potential to turn a losing hand into a break-even or winning hand.

The probability of catching all 4 bonus cards is easily computed, as the only choice to be made is what the fifth card is. There are 48 possibilities, so the probability is

$$\frac{48}{2,598,960} = \frac{1}{54,145}.$$

This value is useful as we find the probability of catching a single winning combination: either the $A\spadesuit$ $J\spadesuit$ or $A\clubsuit$ $J\clubsuit$. The probability of a hand containing the $A\spadesuit$ and $J\spadesuit$ is

$$p = \frac{\binom{50}{3}}{\binom{52}{5}} = \frac{5}{663}.$$

This is also the probability of a hand containing the $A\clubsuit$ and $J\clubsuit$, so the probability of a hand with exactly one of these pairs is

$$2p - \frac{1}{54,145} \approx .0151,$$

about 1 in 66.

How much did Blackjack Power add to the player return? With no wager required to activate this game feature, Blackjack Power has a positive expectation of

$$E = (1) \cdot \frac{5}{663} + (1) \cdot \frac{5}{663} + (50) \cdot \frac{1}{54,145} \approx .0160,$$

for an additional 1.60% return to the player. This raises a 97.3% machine to one that returns 98.9% of players' wagers.

Double Up

Some video poker machines offer the *Double Up* bet as a bonus wager following a player win. If a player chooses this bet, 5 cards are dealt to the screen. One is turned face up as the house card, and the player chooses one of the others. The higher card wins: if the player's card is higher, his winnings on the previous poker hand are doubled; if the house card is higher, the player loses his winnings. If the cards tie in rank, the bet pushes. The bet may be repeated after a player win, depending on the machine.

Under the reasonable assumption that the house card is chosen randomly from among the 5 dealt cards, the house and player cards each have equal chance of being the higher card. Denote the player's winnings by X; the expected value of Double Up is then

$$E = (X) \cdot \frac{1}{2} + (-X) \cdot \frac{1}{2} = 0.$$

This makes Double Up a fair bet with zero house advantage. With that understood, how does a casino make money by adding this option?

It's simple: the wager is only offered after a player win. When players lose, they are not given a second chance to match cards and retrieve the money they've just lost. The opportunity to repeat Double Up after winning gives the casino another 50/50 shot at winning all of a player's accumulated winnings: only 1 gambler in 4 will win twice in a row.

Hopscotch

Possibly the most lucrative video poker bonus was *Hopscotch*, which offered huge payoffs for certain dealt skip straights (page 161). A player dealt a sequential $2468T$, forward or backward, received a 200 for 1 payoff, and a sequential $2468x$ in either order, with any card replacing the x, carried a 100 for 1 payoff [49].

Since the order of the cards matters on a Hopscotch machine, our denominator is ${}_{52}P_5$ rather than $\binom{52}{5}$. The probability of a sequential $2468T$ hand is

$$\frac{4^5}{{}_{52}P_5} \cdot 2 \approx 3.2834 \times 10^{-6}.$$

This probability can also be calculated card-by-card as

$$\frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} \cdot \frac{4}{49} \cdot \frac{4}{48} \cdot 2,$$

illustrating the need to use ${}_{52}P_5$. The 4-card sequential skip straight $2468x, x \neq T$, has probability

$$\frac{4^4 \cdot 44}{{}_{52}P_5} \cdot 2 \approx 3.6117 \times 10^{-5},$$

where the factor of 44 accounts for the fact that the 5th card cannot be a 10 that would win 200 for 1 instead of 100 for 1.

A 9/6 Jacks or Better machine offering the Hopscotch bonus thus had a return percentage of 194.10%, giving the max-coins player a *94.10%* advantage. While the events triggering the bonus were rare, they paid off much more highly than their scarcity justified when the rest of the pay table was considered.

8.5 Exercises

Answers to starred exercises begin on page 339.

8.1.* On early mechanical poker machines, the placement of cards on the 5 reels could be manipulated to limit the operators' risk by reducing the chances of the most lucrative hands. Figure 8.5 shows the assignment of cards to reels on a "so-called honest machine" [138]. As noted on page 285, the $T\spadesuit$ and $J\heartsuit$ are omitted from these reels.

$T\clubsuit$	$3\diamondsuit$	$2\spadesuit$	$Q\heartsuit$	$9\heartsuit$
$4\heartsuit$	$Q\spadesuit$	$J\spadesuit$	$6\heartsuit$	$A\heartsuit$
$J\clubsuit$	$K\diamondsuit$	$4\spadesuit$	$7\heartsuit$	$K\spadesuit$
$8\clubsuit$	$5\clubsuit$	$6\spadesuit$	$2\clubsuit$	$2\heartsuit$
$7\diamondsuit$	$A\clubsuit$	$3\heartsuit$	$6\clubsuit$	$5\spadesuit$
$Q\diamondsuit$	$5\heartsuit$	$A\diamondsuit$	$T\heartsuit$	$J\diamondsuit$
$3\clubsuit$	$9\clubsuit$	$5\diamondsuit$	$T\diamondsuit$	$3\spadesuit$
$K\clubsuit$	$8\diamondsuit$	$K\heartsuit$	$4\diamondsuit$	$6\diamondsuit$
$9\spadesuit$	$4\clubsuit$	$7\clubsuit$	$A\spadesuit$	$Q\clubsuit$
$2\diamondsuit$	$7\spadesuit$	$9\diamondsuit$	$8\heartsuit$	$8\spadesuit$

FIGURE 8.5: Reels from a late 19th-century poker machine.

Assume that all 5-card combinations of symbols on reels are equally likely. Find the probability of the following hands.

a. Royal flush.

b. Four of a kind.

c. Club flush.

d. Heart flush.

8.2.* Figure 8.6 shows a second arrangement of cards on reels, one described as "one of the worst types of the crooked slot machines" [138]. Once again, the $T\spadesuit$ and $J\heartsuit$ are missing.

4♠	7♡	J♠	8♠	7♣
9♠	K♢	6♡	4♡	Q♠
8♢	2♡	K♡	2♠	2♣
K♣	3♢	T♢	5♣	5♢
3♣	Q♡	6♠	9♡	J♣
Q♢	4♣	8♡	6♣	Q♣
8♣	2♢	T♡	A♡	6♢
7♢	T♣	K♠	A♢	3♠
A♠	5♡	3♡	9♢	J♢
4♢	A♣	9♣	7♠	5♠

FIGURE 8.6: Alternate reels from a late 19th-century poker machine.

Repeat Exercise 8.1 for this reel configuration.

8.3. One reason why the reel arrangement in Exercise 8.2 was judged to be crooked was that two reels were locked together and spun in tandem instead of spinning independently, so that the number of possible poker hands was reduced to 1000. Assuming that the linked reels pair up cards in the order shown above (so, for example, linking reels 3 and 4 would mean that the J♠ and 8♠ would always come up together), identify two reels that, if locked together, make it impossible for the player to spin a royal flush or a straight flush.

8.4.* Successive hands of video poker are independent. What is the probability that at least one of the 5 cards initially dealt to a video poker hand reappears as one of the first 5 cards dealt to the next hand?

8.5.* In Jacks or Better video poker, find the probability of drawing a royal flush when discarding 4 cards and holding an ace or face card.

8.6.* What is the Poisson probability of getting at least 1 royal flush in 20,000 hands of Jacks or Better?

8.7.* A November 1995 promotion at the Texas Station Casino in North Las Vegas doubled the payoff, up to 125 coins, on natural four of a kinds on all video poker machines [114]. After losing millions of dollars, the casino canceled the promotion early. Using Table 8.3, find the payback percentage for 9/6 Jacks or Better with this promotion in force.

8.8.* Line 16 of Table 8.16 calls for the player to draw 2 cards to an inside straight flush. Suppose that the 3 cards are the 7,8, and 10 of hearts, which can be completed to a straight flush in 2 ways.

a. Find the probability of completing the straight flush.

b. Find the probability of drawing exactly 1 of the 2 cards needed to complete the straight flush.

8.9.* In a multi-line Jacks or Better game, suppose that you are dealt ♣$AKQT$ $\diamondsuit J$ and discard the $J\diamondsuit$, breaking up your dealt straight in pursuit of a royal flush as Table 8.16 directs. Find the probability of receiving 2 royal flushes if the game has

a. 3 lines.

b. 5 lines.

c. 100 lines.

8.10.* In Example 8.14 (page 322), an initial 2-pair hand at Triple Play Jacks or Better saw 2 of the 3 hands improve to a full house. Find the probability that two 2-pair hands become full houses while the third remains two pairs.

8.11.* Find the number of hands necessary to have a 90% chance of drawing at least one royal flush in Deuces Wild poker and the Poisson approximation to this value.

8.12.* How many ace-high hands containing the $A\bigstar$ are possible in Five Aces?

8.13.* When drawing 1 card to the 4-card straight flush $\diamondsuit 6789$, is the chance of completing the straight flush higher in Jacks or Better or in Five Deck Poker?

8.14.* Find the probability of a dealt full house in Five Deck Poker. How does this compare to the probability of a dealt full house in a poker game using a standard deck?

8.15.* Consider the Five Deck Frenzy hand

$$A\spadesuit\ A\spadesuit\ A\spadesuit\ A\spadesuit\ A\diamondsuit.$$

Table 8.21 asserts that the correct play is to discard the $A\diamondsuit$ and go for 5 $A\spadesuit$s, but this requires a deep trust in the mathematics of expected value since it requires sacrificing a hand with a guaranteed value of 275 coins with a maximum bet.

a. Find the probability that the finished hand after drawing is lower than the unsuited 5 of a kind that is being broken up for the draw.

b. Calculate the expected return when drawing a card, assuming the minimum jackpot of $200,000 and a maximum bet of 5 quarters.

8.16. Find the probability of a dealt two-pair hand in Five Deck Frenzy that is not a flush.

8.17. Another video poker bonus feature is *Stop Sign Poker*, which was added to 8/5 Jacks or Better games, raising the payback to 99.0% [49]. Stop Sign Poker pays a 1 for 1 bonus if the initial hand contains 4 diamonds. If the player draws a fifth diamond, the payoff rises to 2 for 1. Find the probability of a dealt hand with 4 diamonds.

8.18.* *Pay the Aces* was a video poker variation that offered bonus payments if the initial dealt hand contained 1 or more aces and no face cards. The Pay the Aces pay table is shown in Table 8.25.

TABLE 8.25: Pay the Aces pay table for the initial hand. Payoffs are per coin wagered, and all hands require 0 face cards to win.

Number of aces	**Payoff**
1	1–1
2	2–1
3	50–1
4	1500–1

Find the probability of each of the 4 outcomes.

8.19.* *Star Poker* is a video poker game introduced in the early 1990s in Atlantic City [51]. The base game was a 6/5 Jacks or Better game in which one card of each rank from 10 through ace was selected at random and tagged with a star before each new hand. A player dealt or drawing into a royal flush when playing max coins consisting of 5 starred cards received a progressive "super jackpot" that started at $200,000; a royal flush with 4 starred cards and max coins played won a smaller progressive jackpot beginning at $2000. All other royals paid 5000 for 1.

a. Find the probability that the 5 starred cards are *not* all of one suit, thus making the progressive jackpot impossible to win before any cards are dealt.

b. Find the probability that exactly 4 of the 5 starred cards are all of one suit.

c. Optimal strategy for the base game will yield a royal flush with probability $1/35,000$. Find the probability of winning the super jackpot.

Appendix A

Elementary Probability Formulas

Table A.1 collects some of the more useful probability and counting rules that are used in the text.

TABLE A.1: Common probability formulas

Complement Rule	$P(A^C) = 1 - P(A).$
Subset Rule	If $B \subset A$, then $P(B) \leqslant P(A).$
Permutations	${}_nP_r = \frac{n!}{(n-r)!}.$
Combinations	$\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}.$
First Addition Rule (mutually exclusive events)	$P(A \text{ or } B) = P(A) + P(B).$
Second Addition Rule	$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$
Multiplication Rule (independent events)	$P(A \text{ and } B) = P(A) \cdot P(B).$
Conditional Probability	$P(B\|A) = \frac{P(A \text{ and } B)}{P(A)}.$
General Multiplication Rule	$P(A \text{ and } B) = P(A) \cdot P(B\|A).$

In referring to this table, note that the First Addition Rule applies only to mutually exclusive events, while the Second Addition Rule applies to all events. Similarly, the first Multiplication Rule in the table requires that the events A and B be independent; the General Multiplication Rule applies in all cases.

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Answers to Selected Exercises

Chapter 1

Exercises begin on page 51.

1.1. $^{17}\!/_{44} \approx .3864$.

1.2a. 61,776.

1.2b. 2592.

1.3. No.

1.4. $\dfrac{4 \cdot \binom{13}{3}}{\binom{52}{3}} \cdot \dfrac{3 \cdot 10}{\binom{52}{2}} \approx .0012$.

1.5. $^{11}\!/_{221} \approx .0498$.

1.6. \$32.34.

1.7a. $\dfrac{1}{\binom{50}{3}} \approx 5.102 \times 10^{-5}$.

1.7b. +10.5¢.

1.7c. $\sigma^2 = 1275.45\ \$^2$.

1.9. 848.

1.10a. 0.

1.10b. 144.

1.11. .2255.

1.12. .0030.

1.13. $P(5 \text{ of a kind}) \approx .4994$.

1.14. $^{117}\!/_{238} \approx .4916$.

Chapter 2

Exercises begin on page 73.

2.1. $^{16}\!/_{37}$.

2.2. 3.0341×10^{-9}.

2.3. .0348, .0903.

2.5. 1,371,888.

2.6. 271,392 and 270,688. The difference comes from 7-card hands that contain 3 aces and 3 kings. These will be played as aces full rather than kings full.

2.7. There are more straights containing a 10. This is because a 6- or 7-card straight with both a 5 and a 10 will play as a straight containing a 10 when the hand is reduced to 5 cards.

2.8. $1/425$.

2.9. $373/5525 \approx .0675$.

2.10. $32/1105 \approx .0290$.

2.11. .5886.

2.14a. $P(x) = 1 - \dfrac{\binom{32+x}{4}}{\binom{42}{4}} - \dfrac{\binom{10-x}{1} \cdot \binom{32+x}{3}}{\binom{42}{4}}$.

2.14b. $x \leqslant 2$.

2.15. .0029

2.16. 1.4441×10^{-5}.

2.17. .0081.

Chapter 3

Exercises begin on page 105.

3.2. $16/47$.

3.3a. $9/47$.

3.3b. 2.4226×10^{-5}.

3.3c. 4.

3.5a. $4/47$.

3.5b. .3822.

3.5c. .4109.

3.5d. .1688.

3.6a. .1036.

3.6b. .0851.

3.7. .0379.

3.9. 6.8524×10^{-6}.

3.10. .0173.

3.11. .0996.

3.12a. $P(X = k) = \dfrac{\binom{12}{k} \cdot \binom{40}{5-k}}{\binom{52}{5}}$.

3.12b. .2532.
3.12c. $^{15}/_{13} \approx 1.1538$.
3.13. $^{1}/_{595}$.
3.14. 7.959×10^{-5}.
3.15. 8.58.
3.16a. $^{5}/_{663} \approx .0075$.
3.16b. $E = -.2459$; 24.59%.
3.17a. $^{5}/_{663}$; 13.12%.
3.17b. .0067. The player expectation is $E = -.0035$, so the hustler's edge is .175%.
3.18. .0013.
3.19. $20^{5}/_{6}$.
3.20a. 12.5.
3.20b. .2648.
3.20c. 20.56%.

Chapter 4

Exercises begin on page 150.
4.3. $^{6}/_{1225} \approx .0049$.
4.4. $^{1}/_{39}$.
4.5. $^{1}/_{990}$.
4.6. 15 outs. $p \approx .5412$. The Rule of Four approximation is $p \approx .6000$.
4.7. 6.2794×10^{-4}.
4.8a. $^{10}/_{21}$.
4.8b. 1.5391×10^{-5}.
4.8c. $6500.
4.9. 100%. (For details, see [11], pp. 337 and 461.)
4.10. .0965.
4.11. $^{32}/_{175} \approx .1829$.
4.12a. .0084 in hold'em. .0058 in Short Deck.
4.12b. .0948.
4.13. 7812.
4.14. $^{96}/_{187} \approx .5134$.
4.16. $N = 11$ works acceptably well.
4.18. .1055.
4.19. .0032.
4.20. .5032.
4.22. 144.
4.24. .1600; .1681.
4.25. $\dfrac{D^2}{3} \cdot p(1-p)$.

4.26. –$.0045.
4.27. 1.96%.

Chapter 5

Exercises begin on page 183.

5.1. Robin wins, since 3 of a kind beats a 3-card flush when 3 cards constitute a completed hand.

5.3. $1/2548 \approx 3.9246 \times 10^{-4}$. This type of tiger beats a full house but loses to four of a kind.

5.4. 4080.
5.5. 28,561.
5.6. 17,160.
5.7. .0062.
5.9. 2040.
5.13. 1344.
5.16a. 780.
5.16b. 1560.
5.16c. 25,740.
5.16d. 29,928.
5.18. 228,488.
5.20. $2/771,101 \approx 2.5940 \times 10^{-6}$.

Chapter 6

Exercises begin on page 204.

6.1. .0611.
6.2. .0090.
6.3. 9.
6.4. 960.
6.5. 2.4664×10^{-5}.
6.6. $4/11 \approx .3636$.
6.7. 14,417,920.
6.8. .2500.

Chapter 7

Exercises begin on page 278.

7.1a. .0399.

7.1b. $1/221$.

7.1c. With pocket 8s: 3.6938×10^{-5}. Without pocket 8s: 9.2344×10^{-5}.

7.1d. 5.92%.

7.2a. $P(k \text{ 8s}) = \dfrac{\binom{4}{k} \cdot \binom{48}{9-k}}{\binom{52}{9}}$.

7.2b. .8661.

7.2c. 14.60%.

7.3. 7.24%.

7.4. 12.94%.

7.5. The HA is −.06%, which is sufficiently close to 0 to call this fair.

7.7. .0037.

7.8. $216,765.27.

7.9. 73.68%.

7.10. No. The expected value of 223 is negative.

7.11. $1134/1176 \approx .9643$.

7.12. 1.1353×10^{-6}.

7.13. The HA is −0.20%; the *player* has a small edge.

7.14. 73.5%.

7.15. $P(\text{Dealer } AA) = 1/17 > 1/221$.

7.16. .1522.

7.17. .0541.

7.18. $1/595 \approx .0017$.

7.19. $1/35,700 \approx 2.8011 \times 10^{-5}$.

Chapter 8

Exercises begin on page 329.

8.1a. $1/10,000$.

8.1b. $9/1000$.

8.1c. $1/250$.

8.1d. $3/1000$.

8.2a. $1/10,000$.

8.2b. $1/500$.

8.2c. $9/1250$.

8.2d. 0.
8.4. .4098.
8.5. 5.6065×10^{-8}.
8.6. .3935
8.7. 105.43%.
8.8a. .0019.
8.8b. .1221.
8.9a. .0013.
8.9b. .0042.
8.9c. .2723.
8.10. .0199.
8.11. 64 hands. The Poisson probability is .8988.
8.12. 126,208.
8.13. Jacks or Better.
8.14. .0042, approximately triple the probability of a full house dealt from a standard deck.
8.15a. $48/51 \approx 94.1\%$.
8.15b. $3947.79.
8.16. .0691.
8.17. .0107.
8.18. Let k be the number of aces dealt with no face cards. We have

k	$P(k)$
1	.0907
2	.0165
3	9.6919×10^{-4}
4	1.3851×10^{-5}

8.19a. 93.75%.
8.19b. $15/256$.
8.19c. $1/36,000,000$.

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