MAXWELL WITHOUT TEARS—A Fresh Look at His Infamous Equations

By H. Paul Shuch, N6TX
Pennsylvania College of Technology
One College Avenue
Williamsport, PA 17701

Introduction

Let's face it, nobody likes math. Not you, not me, not even the mathematicians. It's what they do for a living. But do you think for a minute they spend their weekends sitting around the back porch, their feet up, a gin and tonic in hand, solving equations? Guess again!

Yet every now and then, an idea comes along that is so important that it needs to be expressed in the elegant simplicity of mathematics. A favorite example is the duality between energy and matter. And though countless books have been written on the topic, you and I still remember the basics best by Einstein's simple equation, $E = MC^2$.

Another case in point, and the topic of this paper, is the propagation of electromagnetic radiation. We can fill volume after volume with convoluted descriptions of how radio waves travel, how antennas work, how transmission lines function—I know, I've written some of those books. Or we can summarize the whole complicated mess by the four simple equations written by James Clerk Maxwell, a Scottish physics professor at Cambridge University, back in 1864.

Maxwell's equations form the basis of our understanding of radio propagation, microwaves, optics, antenna theory, troposcatter, moonbounce, and hold the innermost secrets of DX itself. Except who the heck understands them? Not you, not me, not even the mathematicians.

$$\oint_C E \cdot d\mathbf{l} = -\iint_A \frac{\partial B}{\partial t} \cdot d\mathbf{S}$$

$$\oint_C \frac{B}{\mu} \cdot d\mathbf{l} = \iint_A \left(J + \varepsilon \frac{\partial E}{\partial t} \right) \cdot d\mathbf{S}$$

$$\oiint_A \varepsilon E \cdot d\mathbf{S} = \iiint_V \rho \, dV$$

$$\oiint_A B \cdot d\mathbf{S} = 0$$

Fig 1—Maxwell's integral equations.

The Infamous Equations

Maxwell's equations, whether expressed in integral (Fig 1) or differential (Fig 2) form, describe the behavior of electromagnetic waves. From them we can gain an understanding of wave velocity, direction, polarization, reflection, refraction, diffraction, absorption, attenuation and phase. They are a shorthand notation for how waves work.

More important, Maxwell's equations tell us that radio waves, microwaves, ultraviolet, infrared, visible light, X-rays, gamma rays, and cosmic rays are all the same stuff, differing only in wavelength. And whether emanating from sunlight, satellite or searchlight, all electromagnetic waves behave fundamentally alike.

$$\nabla \cdot E = \rho / \varepsilon$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu \delta E + \mu \varepsilon \frac{\partial E}{\partial t}$$

Fig 2—Maxwell's differential equations.

Free-Space Simplification

Look again at Fig 2, Maxwell's equations in differential form. Go ahead, it won't hurt much. If they seem complex, it's partly because these are general equations, which describe propagation of electromagnetic waves through any medium. Let us here limit our discussion to waves traveling through free space (the stuff in which radio waves used for communication spend most of their time.) We can now simplify the equations considerably, as indicated in Fig 3.

What exactly do we mean by free space? Generally, we mean vacuum, the absence of matter. One of the most

$$\nabla \cdot E = \rho$$

$$\nabla \cdot B = 0$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla \times B = J + \frac{\partial E}{\partial t}$$

Fig 3—Maxwell's differential equations simplified for waves in free space.

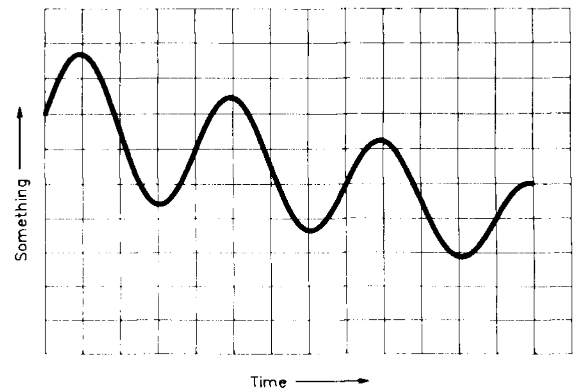

Fig 4—A graph of something varying over time (also known as a function).

spectacular contributions of Maxwell's equations is that they lead us to an understanding that waves can travel through nothingness.

Early radio experimenters had trouble understanding this. They knew what happened to ripples in water when you took the water away, and concluded that all waves needed a medium in which to propagate. For radio waves, they invented the mythical "aether" as a propagation medium.

Researchers spent years looking for the "aether." Only they never found it. Because it doesn't exist. Electromagnetic radiation is pure energy, and Maxwell's equations show that energy can (and does) exist absent any supporting medium. Yes, even in the empty depths of interstellar space. How else would starlight reach us?

When we demonstrate electromagnetic radiation in the laboratory, we generally do a pretty poor job of duplicating free space. On the bench, our signals travel not through vacuum at all, but rather through that mixture of mostly nitrogen, a little oxygen, and various trace elements, which raises a column of mercury 29.92 inches at sea level.

Why don't we surround our waves with vacuum in the laboratory? The problem is how to contain it. Even should we manage to cram a box full to the top with those little vacuum particles, should the box leak, the stuff gets all over the workbench. It makes a mess. Have you ever tried to clean up spilled vacuum? There's really only one way—with a vacuum cleaner.

In reality, the waves care little about our atmosphere. Air is so nearly electrically inert as to have almost no discernible effect upon electromagnetic wave propagation. In truth, air does slow down the waves about six hundredths of a percent. Hardly enough to hassle. So the equations in Fig 3, simplified as they are for vacuum, do a very credible job of describing wave propagation in the laboratory, in air, as well.

The Satanic Calculus

I've borrowed that title (and some of the ideas which follow) from Dr. Bill Rosenthal, Professor of Mathematics at Ursinis College. All of you who got "A's" in calculus can skip the next few sections.

Well, I'm glad to see you're all still here. Calculus is the mathematics of functions, so let's take a look at a function (see Fig 4). Here's a graph of something changing over time. It could be an electrical waveform viewed on an oscilloscope (the electronicker calls it "damped oscillation with a steadily decreasing dc offset"), or perhaps a graph of how our body temperature varies throughout the day. No matter. Calculus is merely a set of tools for describing any function mathematically.

Integrally Speaking

One of those tools is the integral, closely associated with the "area under the curve." If I hand Fig 4 to six different students, and ask them to come up with a method for determining the area under the curve, I'm likely to get a half dozen different approaches. Let's examine them.

Student number one knows his calculus. He writes an equation for the function, and meticulously applies boring and repetitive memorized steps, until he has extracted an integral. His result is entirely correct, and altogether

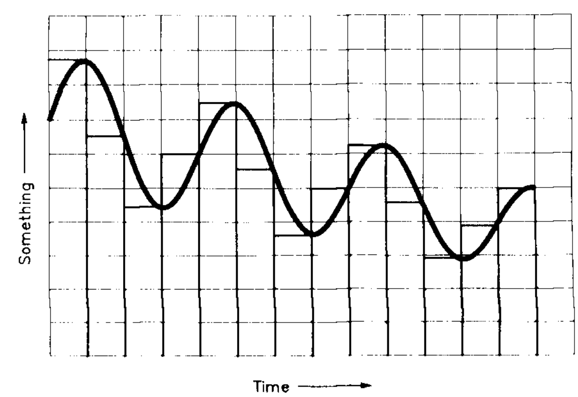

Fig 5—Integration by summing rectangles.

uninteresting. It demonstrates great mathematical facility, but no understanding whatever of the nature of the function. Next!

Our second student is fluent in geometry. She divides the space under the curve into a whole bunch of rectangles (Fig 5) and, since she knows how to compute the area of a rectangle, has no trouble whatever in summing them to approximate the whole.

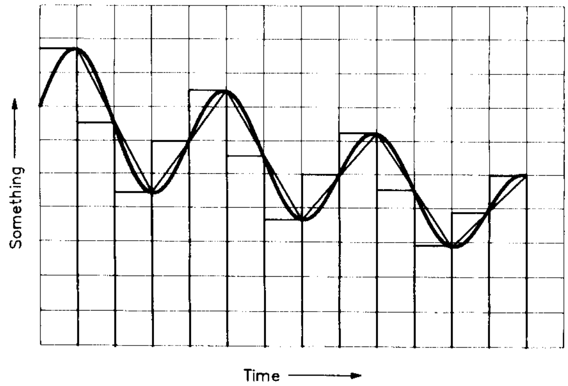

Fig 6—Including the triangles reduces the error.

Her classmate, a Pythagorean by nature, notices in her approximation a slew of left-over triangles (Fig 6). This student knows how to compute the area of a triangle, and adds (or subtracts) the extra pieces for an even closer estimate.

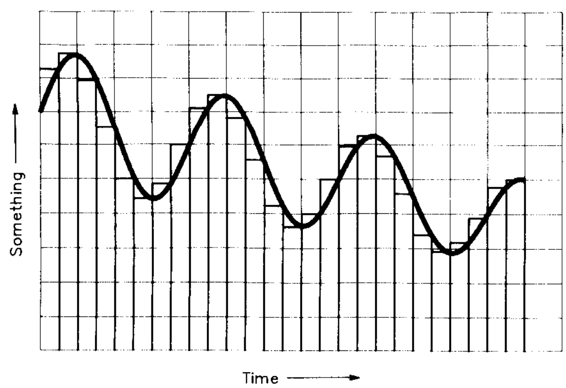

Fig 7—Infinitesimals further reduce the error.

Student number four understands infinitesimal calculation. She reasons that if the rectangles are made very skinny (Fig 7), the errors caused by the left-over pieces begin to diminish. Her result is rather precise, but still demonstrates little understanding of the underlying function.

The fifth student is satisfied with a logical approximation. He runs a straight edge along the wavy curve (Fig 8),

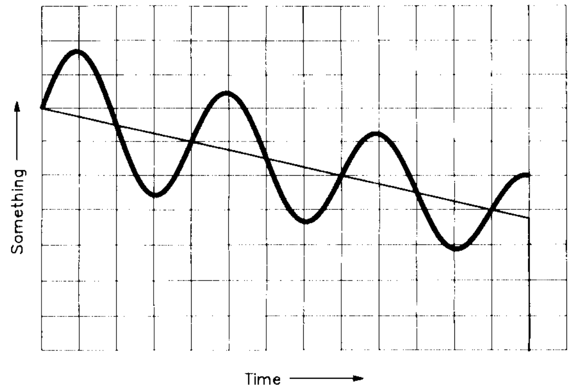

Fig 8—Our function as cyclical variations around a trend line.

with roughly as much "stuff" above as below the line. It's easy to calculate the area of the resulting trapezoid. More important, this student is the first to recognize that our function involves cyclical variation about a trend line.

The last student, a natural born technician, prefers a manipulative solution. Gluing the graph paper to a sheet of plywood, she uses a saber saw to carefully cut out the shape, which she then weighs. By also weighing a rectangular piece of the same plywood (the area of which is of course known), she easily determines the area under the curve. She also understands, better than her classmates, that an integral is a measure of how much "stuff" a function encloses.

Deriving a Derivative

Another tool for evaluating a function is its derivative, or rate of change. This is generally equated to the slope of lines drawn tangent to the curve at various points, as shown in Fig 9. Note, for example, that the curve exhibits a positive slope at point A (the dependent variable is increasing with time), a negative slope at B, and a slope

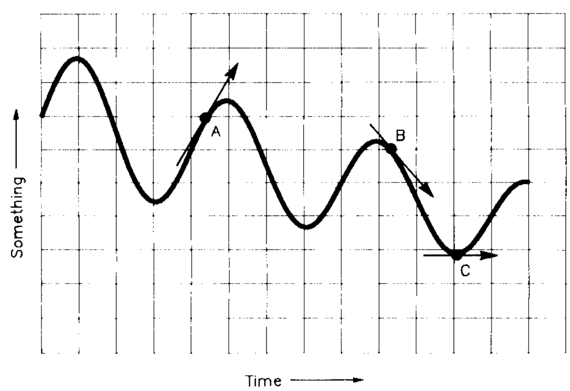

Fig 9—Derivative of a function relates to instantaneous slope.

of zero (a horizontal tangent line) at C. This latter characteristic, zero slope, is typical of functions at their minimum or maximum value, thus derivatives are a powerful tool for locating global or local maxima and minima.

But all of this is entirely too mathematical. Let's make the derivative more intuitive, by flying an airplane along the surface of our now familiar function. As we proceed from left to right along the page, we apply wing flaps, elevator trim, and stick pressure, as well as varying throttle, fuel mixture, and propeller pitch, as necessary, to trace out our function in the sky. Notice that the aircraft's pitch attitude (the angle which the nose makes with respect to the horizon) is ever changing. Its rate of change (time derivative) fully describes the given function.

One Thing at a Time

Of course, our pilot flying Fig 9's has his hands full. What with simultaneous changes in stick pressure, elevator trim, airspeed, mixture, throttle, prop and flaps, it's hard to know just which input variable caused the observed response. Which is true of most phenomena we measure in the laboratory. The more variables we have to deal with, the harder it is to relate cause to effect accurately.

Enter the partial derivative, a mathematical tool for measuring constrained rates of change. Its basic tenet is to hold all independent variables constant except one, and observe how the function responds to changes in that one parameter alone. What will happen to the airplane's pitch attitude if I vary, for example, throttle setting only, leaving everything else alone? Partial derivatives answer just such questions.

The accepted notation for partial derivatives is a fraction, in which the numerator represents the dependent, and the denominator the independent variable of a function. Each is preceded by the symbol δ, the Greek lower-case letter delta. For example, $\delta B/\delta t$ represents the partial derivative of current with respect to time, also known as the partial time derivative of current. We encounter precisely this notation in Maxwell's equations.

Now that we have reviewed a few of the basics of calculus, we are nearly ready to do what Maxwell did, and use them to describe electromagnetic waves. But first, we have to add just a couple more tools to our analytical workbench, and they come to us from the discipline of Vector Calculus.

Curl (In or Out, Your Choice)

If Robin of Loxley truly was the greatest marksman of twelfth century England, it might be in part because he imparted curl to his arrows. In the cinema, he's always shown drawing back on his bow, and then twisting his wrist slightly. The arrows fly straighter because they rotate about their longitudinal axis. You can see for yourself—just watch the feathers!

In vector calculus, curl brings to mind things rotating, or swirling, around. It is related to angular velocity, and has to do with a line integral around a closed path. Since, for example, we can easily visualize a magnetic field existing *around* a current carrying conductor, you can understand why Maxwell included curl in his equations.

In equations, curl is indicated by the operator del (∇, an upside-down capital Greek letter delta). It involves the vector algebra cross-product. Thus, for example, "$\nabla \times E$" would be pronounced "the curl of E," and has to do with the rotation of an electric field vector. When we operate with ∇ by crossing it into a vector function, we get the *curl* of that function.

Diverge Away!

When I first studied fields and waves, I had great difficulty visualizing expanding and collapsing fields. The picture became clear for me when I chanced to see a classmate preparing decorations for a dorm party—he was blowing up balloons. When they inflated, both the enclosed volume and the surface area increased. And when they later deflated, the opposite occurred.

We can illustrate divergence, a scalar function of position, in terms of balloons deflating. Formally, divergence is defined for a vector function as the ratio of the surface integral to the volume enclosed, as the volume shrinks to zero about some point. Functionally, divergence tells us how dense the "stuff" in a field becomes as the field collapses.

In equations, divergence is also indicated by the operator del (∇, the same upside-down capital Greek letter delta we used for curl). It involves the vector algebra dot-product. Thus, for example, "$\nabla \cdot E$" would be pronounced "the divergence of E," and has to do with a collapsing electric field vector. When we operate with ∇ by dotting it into a vector function, we get the *divergence* of that function.

Credit Where It's Due

We know today (thanks in no small part to Maxwell's work) that an electromagnetic wave propagating through free space is pure energy, composed of mutually orthogonal (that is, at right angles to each other) electrostatic and magnetic force fields. Maxwell was hardly the first person to posit the existence of such waves. They were the topic of lively discussion by the likes of Gauss, Ampere, Faraday and others a generation prior. Nor did Maxwell generate and detect such waves in the laboratory. That feat is credited to Hertz a generation later. Maxwell's great contribution to technology lies in his recognition that, as far as energy propagation is concerned, electricity and magnetism are interchangeable.

Not to minimize his individual efforts, but perhaps Maxwell's greatest contribution to radio science comes from his recognizing the importance of the work of others. He synthesized several theories about electricity and magnetism into the four equations we know and love so well. Let's take another look at these equations in simplified (free-space) differential form, as seen in Fig 3.

Those Dreaded Equations Again!

Maxwell's first equation reads: "the divergence of an electric field vector is equal to the corresponding electric charge." This is a restatement of Gauss' Law for Electricity. We know that an electrically charged particle will have an electrostatic field around it. This equation describes the shape of the electric field caused by an electric charge.

The second equation states: "the divergence of a magnetic field vector equals zero." This is Gauss' Law for Magnetism, which describes the shape of the magnetic field surrounding a magnet. You can actually see the shape of this field (at least in two dimensions), by covering a magnet with paper, and shaking on some iron filings. The zero on the right hand side of the equation indicates you cannot have a magnetic monopole (that is, a North pole without a South, or vice-versa).[1]

The third equation tells us that we can create an electrical field by varying a magnetic field over time. We do this in electrical generators, every time we move a magnet past a wire (or a wire past a magnet—it works both ways). The equation reads "the curl of an electric field vector, and the partial time derivative of the associated magnetic field vector, sum to zero." The zero sum indicates that energy is conserved, and the equation is also known as Faraday's Law.

Maxwell's final equation can be read: "the curl of a magnetic field vector equals the sum of the electric current vector plus the partial time derivative of the electric field vector." If you think this sounds like another conversion in which energy is conserved, you're beginning to catch on! A magnetic field can be created by either a current, or by an electric field which varies over time. Or by a combination of the two. This relationship, also known as Ampere's Law, describes the shape of just such a field.

So What Do They Mean?

Let's consider how radio waves propagate. At the transmitter, we generate an alternating electric charge (that is, an ac voltage) which we apply to a wire (the transmit antenna). A time varying electrostatic field now surrounds the wire, in accordance with Maxwell's first equation. As the electrostatic field expands and collapses, it induces in front of it an expanding and collapsing magnetic field, the shape of which is described by Maxwell's fourth equation. This time varying magnetic field will induce ahead of it an expanding and collapsing electrostatic field, as described by Maxwell's third equation.

And so the wave propagates: expanding and collapsing electrostatic field, inducing ahead of it an expanding and collapsing magnetic field, which gives rise to an expanding and collapsing electrostatic field. Maxwell's third equation, fourth equation, third equation and so forth, until the wave reaches its destination.

Let's suppose the forward propagating wave encounters a nearby magnet enroute. This happens, for example, when space communications signals encounter the large round magnet which we call the earth. Maxwell's second equation tells us the shape of the magnetic field which surrounds a magnet. This field influences the forward propagated signal, and is responsible for the change in polarization (Faraday Rotation) with which moonbouncers are quite familiar.

Eventually, our propagated wave produces an electrical field in the vicinity of a receive antenna. An emf is induced into the conductor by the surrounding electrostatic field (Maxwell's first equation in reverse), and we now have a received voltage to process.

In this over-simplified example, we see that Maxwell's equations account for transmission, reception, antenna operation, propagation and modification of electromagnetic waves. They also enable us to compute such important constants as the speed of light (Maxwell himself did this, in terms of the assumed permittivity and permeability of free space, with results which correlated remarkably well with measured values), and the characteristic impedance of free space (that is, the impedance to which our antennas never quite match!). Not to mention making it quite clear why we can't receive well on our rubber duckies, signals which were transmitted from a horizontally polarized beam.

Of course, we have been dealing thus far with simplified versions of Maxwell's equations, which neglect dielectric and magnetic materials. By going back to the original equations (Figs 1 and 2), we could determine how (and why) electromagnetic waves slow down in the dielectric of coaxial cables or PC boards; why (and by how much) resonant frequency changes when we insert a powdered iron core into an inductor; and how much loss (fortunately, precious little) radio waves suffer when forced to travel through that mixture of mostly nitrogen, a little oxygen, and trace gasses which raises a column of mercury 29.92 inches at sea level.

Sure, you could do all that without Maxwell's equations. But isn't it nice to know there really is a mathematical basis for all we do with radio waves?

Conclusions

Nobody really likes math. Not you, not me, not even the mathematicians. But if we can put our innumeracy aside long enough to take a hard look at Maxwell's equations, we will see that they provide an analytical framework for all of our radio propagation experiments. Maxwell's equations certainly don't make our tropo, E-skip, meteor or EME signals any louder. They simply make them possible.

Acknowledgement

This paper was originally presented at *Microwave Update '91* in Arlington TX, on October 19, 1991, and appeared in the *Proceedings* of that Conference.

References

Hecht, E., *Optics*, second edition [1987], Addison-Wesley, Reading, MA, pp 39-42; 620-622.

Hitz, C. B., *Understanding Laser Technology*, second edition [1991], PennWell Books, Tulsa, OK, pp 14-15.

"Hunting Down the Magnetic Monopole," *Science News*, Vol 140 #14, p 219, October 5, 1991.

Maxwell, J. C., "A Dynamical Theory of the Electromagnetic Field," [1864] Royal Society, London.

Rosenthal, W. E., "A Humanistic Calculus," *AMATYC Review*, Fall 1989, Vol 11 No. 1.

Schey, H. M., *Div, Grad, Curl and All That; An Informal Text on Vector Calculus*, 1st edition [1973], W. W. Norton And Co, NY.

[1]Maxwell notwithstanding, physicists at the Monopole, Astrophysics, and Cosmic Ray observatory (MACRO) at Italy's Gran Sasso National laboratory have reason to believe that fundamental particles with but a single pole may in fact exist, and are currently engaged in a search for the elusive magnetic monopole [*Science News*], October, 5, 1991.